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RESEARCH REPORT

N° 7912

March 2012

Project-Team Geometrica

ISRN INRIA/RR--7912--FR+ENG

ISSN 0249-6399

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Research Report n° 7912 — March 2012 — 32 pages

Abstract: This paper proposes an algorithm to mesh 3D domains bounded by piecewise smooth surfaces. The algorithm handle multivolume domains defined by surfaces that may be non connected or non manifold. The boundary and subdivision surfaces are assumed to be described by a complex formed by surface patches stitched together along curve segments.
The meshing algorithm is a Delaunay refinement and it uses the notion of restricted Delaunay triangulation to approximate the input curve segments and surface patches. The algorithm yields a mesh with good quality tetrahedra and offers a user control on the size of the tetrahedra. The vertices in the final mesh have a restricted Delaunay triangulation to any input feature which is a homeomorphic and accurate approximation of this feature. The algorithm also provides guarantee on the size and shape of the facets approximating the input surface patches. In its current state the algorithm suffers from a severe angular restriction on input constraints. It basically assumes that two linear subspaces that are tangent to non incident and non disjoint input features on a common point form an angle measuring at least 90 degrees.

Key-words: mesh generation, Delaunay refinement, restricted Delaunay triangulation

* Work partially supported by the IST Programme of the EU as a Shared-cost RTD (FET Open) Project under Contract No IST-006413 (ACS - Algorithms for Complex Shapes) and by the European Network of Excellence AIM@SHAPE (FP6 IST NoE 506766).

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Maillage de domaines 3D bornés par des surfaces lisses par morceaux

Résumé : Ce document propose un algorithme pour mailler des domaines 3D bornés par des surfaces lisses par morceaux. L'algorithme traite les domaines sous-divisés. Les domaines sont définis par des surfaces qui peuvent avoir plusieurs composantes connexes et des singularités. Les surfaces de contraintes définissant les frontières du domaine et ses sous-divisions sont des complexes formés de carreaux de surfaces assemblés le long de segment de courbes.

L'algorithme génére le maillage par raffinement de Delaunay et utilise la notion de Delaunay restreint pour approximer les segments de courbes et carreaux de surfaces. L'algorithme produit un maillage de qualité et offre à l'utilisateur un contrôle sur la taille des tétraèdres. Les sommets du maillage produit sont tels que la restriction de leur triangulation de Delaunay, à chaque élément de contrainte (segment de courbe ou carreau de surface), est homéomorphe à cet élément et en constitue une approximation précise. L'algorithme offre aussi des garanties sur la taille et la forme des triangles qui constitue l'approximation des surfaces de contrainte. Dans sa version courante, l'algorithme impose une sévère restriction angulaire aux surfaces de contraintes, supposant que tout paire de sous-espace linéaires tangents en un même point à deux élèments de contraintes non-incidents et non disjoints forment un angle d'au moins 90 degrés.

Mots-clés : génération de maillages, raffinement de Delaunay, triangulation de Delaunay restreinte

1 Introduction

Mesh generation is a notoriously difficult task. Getting a fine discretization of the domain of interest is the bottleneck step of many applications in the area of modelization, simulation or scientific computation. The problem of mesh generation is made even more difficult when the domain to be meshed is bounded and structured by curved surfaces which have to be approximated as well as discretized in the mesh. This paper deal with the problem of generating unstructured tetrahedral mesh for domains bounded by piecewise smooth surfaces. A common way to handle such a meshing problem consists in building first a triangular mesh approximating the boundary surfaces and then refine the discretization of the volumes while preserving the surface approximation. The meshing of surfaces are mostly performed through the highly popular marching cubes algorithm [LC87]. The marching cubes algorithm provides an accurate discretization of smooth surfaces but the output surface mesh generally includes poor quality elements and it fails to recover sharp features. This marching cube may be followed by some remeshing step to improve the shape of the elements and adapts the sizing of the surface mesh to the required density, see [AUGA05] for survey on surface remeshing. Once a boundary surface mesh is obtained, this piecewise linear approximation is substitute to the original surface. The three dimensional mesh is then obtained through a meshing software that either conforms strictly to the boundary surface mesh (see e.g. [FBG96, GHS90, GHS91]) or allows to refine the surface mesh within the geometry of the piecewise linear approximation [She98, CDRR04]. See e.g. [FG00] for a survey on three dimensional meshing. In both cases, the quality of the resulting mesh and the accuracy of the boundary approximation depend highly on the initial surface mesh P.

This paper proposes an alternative to the marching cube strategy. In this alternative, the recovery of bounding curves and surfaces is based on the notion of restricted Delaunay triangulations and the mesh generation algorithm is a multi level Delaunay refinement process which interleaves the refinement of the curves, surfaces and volumes discretization.

Delaunay refinement is recognized as one of the most powerful method to generate meshes with guaranteed quality. The pioneer works of Chew [Che89] and Ruppert [Rup95] handle the generation of two-dimensional meshes for domains whose boundaries and constraints do not form small angle. Shewchuk improved the handling of small angles in two dimensions [She02] and generalized the method to generate three-dimensional meshes for domains with piecewise linear boundaries [She98]. The handling of small angles formed by constraints is more puzzling in three dimensions, where dihedral angles and facet angles come into play. Using the idea of protecting spheres around sharp edges [MMG00, CCY04], Cheng and Poon [CP03] provided a thorough handling of small input angles formed by boundaries and constraints. Cheng, Dey, Ramos, and Ray [CDRR04] turned the same idea into a simpler and practical meshing algorithm.

In three-dimensional space, Delaunay refinement produce tetrahedral meshes free of all kind of degenerate tetrahedra except slivers. Further works [CDE$^+$00, CD03, LT01, CDRR05] were needed to deal with the problem of sliver exudation.

Up to know, little work has been dealing with curved objects. The early work of Chew [Che93] concern the meshing of curved surfaces and in [BOG02], Boivin and Ollivier-Gooch consider the meshing of 2-dimensional domains with curved boundaries. In [ORY05], we proposed a Delaunay refinement algorithm to mesh a 3-dimensional domain bounded by a smooth surface. The algorithm rely on recent results on surface meshing [BO05, BO06]. It involves Delaunay refinement techniques to provide a nice sampling of both the volume and the bounding surface and the notion of restricted Delaunay triangulation to extract, from the Delaunay triangulation of the sample, a piecewise linear approximation of the boundary surface. The present paper extends this mesh generation algorithm to handle 3-dimensional domains defined by piecewise smooth surfaces, *i.e.* patches of smooth surfaces stitched together along 1-dimensional smooth curved segments. The 1-dimensional features are approximated through their Delaunay restricted triangulation and the accuracy of the approximation is controlled by a few additional refinement rules in the Delaunay refinement process. Our meshing algorithm ends up with a controlled quality mesh in which each surface patch and curved segment has a homeomorphic piecewise linear approximation at a controlled Hausdorff distance. The algorithm can handle multi-volume domains defined by piecewise smooth surfaces which may be non connected or non manifold. The only severe restriction on the input features is an angular restriction. Roughly speaking, tangent planes on a common point of two adjacent surface patches are required to make an angle bigger than 90 °. The algorithm rely only on a few oracles able to detect and compute intersection points between straight segments and surface patches or between straight triangles and 1-dimensional curved segments. Therefore it can be used in various situations like meshing CAD-CAM models, molecular surfaces or polyhedral models.

Our work is very closed to a recent work [CDR07] where Cheng, Dey and Ramos proposed a Delaunay refinement meshing for piecewise smooth surfaces. There algorithm suffers no angular restriction but uses topologically driven refinement rules which involve computationally intensive and hard to implement predicates on the surface.

The paper is organized as follows. Section 2 precises the input of the algorithm and provides a few definitions. In particular we define a local feature size adapted to the case of piecewise smooth surfaces. We describe the meshing algorithm in section 3. Before proving in section 5 the correctness of this algorithm, *i.e.* basically the fact that it always terminates, we prove in section 4 the accuracy, quality and homeomorphisms properties of the resulting mesh 4. The algorithm has been implemented using the library CGAL [CGAL]. Section 6 provides some implementation details and shows a few examples. The last section 7 gives some directions for future work, namely to get rid of the angular restriction on input surface patches.

2 Input, definitions and notations

2.0.1 Input

The domain $\mathcal{O}$ to be meshed is assumed to be a union of three dimensional cells whose boundaries are piecewise smooth surfaces *i.e.* formed by smooth surface patches joining along smooth curve segments.

More precisely, we define a regular complex as a set of closed manifolds, called faces, such that :
- any two faces have disjoint interior,
- the boundary of each face is a union of lower dimensional faces of the complex.

We consider a 3-dimensional regular complex whose 2-dimensional subcomplex is formed with patches of smooth surfaces and whose 1-dimensional skeleton is formed with smooth curve segments. Each curve segment is assumed to be a compact subset of a smooth closed curve, and each surface patch is assumed to be a compact subset of a smooth closed surface. The smoothness conditions on curves (resp. surfaces) is to be $C^{1,1}$, *i.e.* to be differentiable with a Lipschitz condition on the tangent (resp. normal) field.

The domain $\mathcal{O}$ that we consider is a union of cells, *i.e.* of 3-dimensional faces, in such a regular complex.

We denote by $\mathcal{F}$ the 3-dimensional regular complex which describe the domain. The set of faces in $\mathcal{F}$ includes a set $\mathcal{Q}$ of vertices, a set $\mathcal{L}$ of smooth curve segments, a set $\mathcal{S}$ of smooth surface patches and a set $\mathcal{C}$ of 3-dimensional cells, such that $\mathcal{F} = \mathcal{Q} \cup \mathcal{L} \cup \mathcal{S} \cup \mathcal{C}$. The domain $\mathcal{O}$ to be meshed is just the union $\mathcal{O} = \bigcup_{F \in \mathcal{F}} F$ of faces in $\mathcal{F}$. For convenience, we also note $\mathcal{F}_2$ the subcomplex of $\mathcal{F}$ formed by the faces of dimension at most 2, and $\bigcup \mathcal{F}_2$ the domain covered by those faces, that is $\mathcal{F}_2 = \mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}$ and $\bigcup \mathcal{F}_2 = \bigcup_{F \in \mathcal{F}_2} F$.

We assume that two elements in $\mathcal{F}_2$ which are neither disjoint nor incident do not form sharp angles. More precisely, denoting by $\mathrm{d}(x, y)$ the Euclidean distance between two points x and y, we assume the following:

Definition 1 (The angular hypothesis)**.** *There is a distance λ_0 so that, for any pair (F, G) of non disjoint and non incident faces in $\mathcal{F}_2$, if there is a point z on $F \cap G$ such that $\mathrm{d}(x, z) \leq \lambda_0$ and $\mathrm{d}(y, z) \leq \lambda_0$, then the following inequality holds:*

$$\mathrm{d}(x, y)^2 \quad \geq \quad \mathrm{d}(x, F \cap G)^2 + \mathrm{d}(y, F \cap G)^2. \tag{1}$$

In the special case of linear faces, the angular hypothesis holds when the *projection condition* [She98] holds. The projection condition states that if two elements F and G of $\mathcal{F}_2$ are neither disjoint nor incident, the orthogonal projection of G on the subspace spanned by F does not intersect the interior of F. For two adjacent planar facets, it means that the dihedral angle must be at least 90°.

2.0.2 Definition of the local feature size

To describe the sizing field used by the algorithm we need to introduce a notion of *local feature size* (lfs , for short) related to the notion of local feature size used for polyhedra [Rup95, She98] and also to the local feature size introduced [AB99] for smooth surfaces.

To account for curvature of input curve segments and surface patches, we first define a notion of *interrelated points*. We use here an idea analogous to the notion of *intertwined* points introduced in [LS03] for anisotropic metric.

Definition 2 (Interrelated points)**.** *Two points x and y of $\mathcal{F}_2$ are said to be* interrelated *if:*

- *either they lie on a common face $F \in \mathcal{F}_2$,*
- *or they lie on non-disjoint faces, F and G, and there exists a point w in the intersection $F \cap G$ such that: $\mathrm{d}(x, w) \leq \lambda_0$ and* $\mathrm{d}(y, w) \leq \lambda_0$.

We first define a feature size $\mathrm{lfs}^{\mathrm{P}}(x)$ analog to the feature size used for a polyhedron. For each point $x \in \mathbb{R}^3$, $\mathrm{lfs}^{\mathrm{P}}(x)$ is the radius of the smallest ball centered at x that contains two non interrelated points of $\bigcup \mathcal{F}_2$.

We then define a feature size $\mathrm{lfs}_{F_i}(x)$ related to each feature in $\mathcal{L} \cup \mathcal{S}$. Let F_i be a curve segment of $\mathcal{L}$ or surface patch in $\mathcal{S}$. We first define the function $\mathrm{lfs}_{F_i}(x)$ for any point $x \in F_i$ as the distance from x to the medial axis of the smooth curve or the smooth surface including the face F_i. Thus defined, the function $\mathrm{lfs}_{F_i}(x)$ is a Lipschitz function on F_i. Using the technique of Miller, Talmor and Teng [MTT99], we extend it as a Lipschitz function $\mathrm{lfs}_{F_i}(x)$ defined on $\mathbb{R}^3$:

$$\forall x \in \mathbb{R}^3,\ \mathrm{lfs}_{F_i}(x) = \inf_{x' \in F_i} \{\mathrm{d}(x, x') + \mathrm{lfs}_{F_i}(x')\}.$$

The local feature size $\mathrm{lfs}(x)$ used below is defined as the pointwise minimum:

$$\mathrm{lfs}(x) = \min\left(\mathrm{lfs}^{\mathrm{P}}(x), \min_{F_i \in \mathcal{F}} \mathrm{lfs}_{F_i}(x)\right).$$

Being a pointwise minimum of Lipschitz functions, $\mathrm{lfs}(x)$ is a Lipschitz function.

3 The mesh generation algorithm

The meshing algorithm is based on the notions of Voronoi diagrams, Delaunay triangulation and restricted Delaunay triangulations which are briefly recalled here.

Let $\mathcal{P}$ be a set of points and p a point in $\mathcal{P}$. The Voronoi cell $V(p)$ of the point p is the locus of points that are closer to p than to any other point in $\mathcal{P}$. For any subset $\mathcal{T} \in \mathcal{P}$ we note $V(\mathcal{T})$ the intersection $\bigcap_{p\in\mathcal{T}} V(p)$. The Voronoi diagram $\mathcal{V}(\mathcal{P})$ is the complex formed by the non empty Voronoi faces $V(\mathcal{T})$ for $\mathcal{T} \subset \mathcal{P}$.

The natural embedding of the nerve of $\mathcal{V}(\mathcal{P})$, includes the simplex $conv(\mathcal{T})$, which is the convex hull $\mathcal{T}$, for any subset $\mathcal{T} \in \mathcal{P}$ that has a non empty Voronoi cell. In non degenerate cases, that is when there is no subset of five or more cospherical points in $\mathcal{P}$, this nerve realization is a triangulation which is called the Delaunay triangulation $\mathcal{D}(\mathcal{P})$ of $\mathcal{P}$.

Let $\mathcal{X}$ be a subset of $\mathbb{R}^3$. We call Delaunay triangulation restricted to $\mathcal{X}$ and note $\mathcal{D}_{|\mathcal{X}}(\mathcal{P})$ the subcomplex of $\mathcal{D}(\mathcal{P})$ formed by faces in $\mathcal{D}(\mathcal{P})$ whose dual Voronoi faces have a non empty intersection with $\mathcal{X}$. Thus a triangle pqr of $\mathcal{D}(\mathcal{P})$ belongs to $\mathcal{D}_{|\mathcal{X}}(\mathcal{P})$ iff the dual Voronoi edge $V(p,q,r)$ has a non empty intersection with $\mathcal{X}$ and an edge pq of $\mathcal{D}(\mathcal{P})$ belongs to $\mathcal{D}_{|\mathcal{X}}(\mathcal{P})$ iff the dual Voronoi facet $V(p,q)$ has a non empty intersection with $\mathcal{X}$.

The algorithm is a Delaunay refinement algorithm that iteratively builds a set of sample points $\mathcal{P}$ and maintains the Delaunay triangulation $\mathcal{D}(\mathcal{P})$, its restriction $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ to the domain $\mathcal{O}$ and the restrictions $\mathcal{D}_{|S_k}(\mathcal{P})$ and $\mathcal{D}_{|L_j}(\mathcal{P})$ to and every facet S_k of $\mathcal{S}$ and every edge L_j of $\mathcal{L}$. At the end of the refinement process, the tetrahedra in $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ form the final mesh and the subcomplexes $\mathcal{D}_{|S_k}(\mathcal{P})$ and $\mathcal{D}_{|L_j}(\mathcal{P})$ are accurate approximation of respectively S_k and L_j. The refinement rules applied by the algorithm to reach this goal use the, hereafter defined, notion of encroachment for restricted Delaunay facets and edges.

Let L_j be an edge of $\mathcal{L}$. For each edge qr of the restricted Delaunay triangulation $\mathcal{D}_{|L_j}(\mathcal{P})$, there is at least one ball, centered on L_i, whose bounding sphere passes through q and r and with no point of $\mathcal{P}$ in its interior. Such a ball is centered on a point of the non empty intersection $L_j \cap V(q,r)$ and called here after a restricted Delaunay ball. The edge qr of $\mathcal{D}_{|L_j}(\mathcal{P})$ is said to be encroached by a point p if p is in the interior of a restricted Delaunay ball of qr.

Likewise, for each triangle qrs of the restricted Delaunay triangulation $\mathcal{D}_{|S_k}(\mathcal{P})$, there is at least one ball, centered on the patch S_k, whose bounding sphere passes through q, r and s and including no point of $\mathcal{P}$ in its interior. Such a ball is called a restricted Delaunay ball (or a surface Delaunay ball in this case). The triangle qrs of $\mathcal{D}_{|S_k}(\mathcal{P})$ is said to be encroached by a point p if p is in the interior of a surface Delaunay ball of qrs.

The refining rules also use the radius-edge ratios of triangles and tetrahedra. The radius-edge ratio of a triangle or of a tetrahedra is the ratio from the circumradius to the length of the shortest edge.

The algorithm takes as input

- the regular complex $\mathcal{F}$ describing the domain to be meshed.
- A sizing field $\sigma(x)$ defined over the domain to be meshed. The sizing field is assumed to be a Lipschitz function such that for any point $x \in \mathcal{F}_2$, $\sigma(x) \leq \mathrm{lfs}(x)$.
- Some shape criteria given by two constant β_2 and β_3. which are upperbounds for the radius-edge ratios of respectively the boundary facets, *i.e.* facets of $\mathcal{D}_{|\bigcup\mathcal{F}_2}(\mathcal{P})$, and the mesh tetrahedra, *i.e.* tetrahedra of $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$.

The algorithm begins with a set of sample points $\mathcal{P}_0$ including $\mathcal{Q}$, at least two points on each segment of $\mathcal{L}$ and at least three points on each patch of $\mathcal{S}$.

At each step a new sample point is added to the current set $\mathcal{P}$ and the algorithm updates the Delaunay triangulation $\mathcal{D}(\mathcal{P})$ and its restrictions $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$, $\mathcal{D}_{|L_j}(\mathcal{P})$ and $\mathcal{D}_{|S_k}(\mathcal{P})$ to respectively the domain $\mathcal{O}$, every edge L_j in $\mathcal{L}$ and every facet S_k of $\mathcal{S}$. The new point is added according to the following rules, where rule R_i is applied only when no rule R_j, with $j < i$, can be applied. Those rules issue calls to sub-procedures, respectively called `refine-edge`, `refine-facet-or--edge`, and `refine-tet-facet-or-edge`, which are described below. The parameters α_1 and α_2 are small constants such that $\alpha_1 \leq \alpha_2 \leq 1$, they will be fixed later.

R1 If, for some L_j of $\mathcal{L}$, there is an edge e of $\mathcal{D}_{|L_j}(\mathcal{P})$ whose endpoints do not both belong to L_j, call `refine-edge`(e).

R2 If, for some L_j of $\mathcal{L}$, there is an edge e of $\mathcal{D}_{|L_j}(\mathcal{P})$ with a restricted Delaunay ball $B(c_e, r_e)$ that does not satisfy $r_e \leq \alpha_1 \sigma(c_e)$, call `refine-edge`$(e)$.

R3 If, for some S_k of $\mathcal{S}$, there is a facet f of $\mathcal{D}_{|S_k}(\mathcal{P})$ whose vertices do not all belong to S_k, call `refine-facet-or-edge`(f).

R4 If, for some S_k of $\mathcal{S}$, there is a facet f of $\mathcal{D}_{|S_k}(\mathcal{P})$ and a restricted Delaunay ball $B(c_f, r_f)$ with a radius-edge ratio ρ_f such that:

R4.1 either the size criteria, $r_f \leq \alpha_2 \sigma(c_f)$, is not met,

R4.2 or the shape criteria, $\rho_f \leq \beta_2$, is not met,

call `refine-facet-or-edge`(f).

R5 If there is some tetrahedron t in $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$, whose Delaunay ball $B(c_t, r_t)$ has the radius edge ratio ρ_t such that:

R5.1 either the size criteria, $r_t \leq \sigma(c_t)$, is not met,

R5.2 or the shape criteria $\rho_t \leq \beta_3$ is not met,

call `refine-tet-facet-or-edge`(t).

refine-edge The procedure `refine-edge`(e) is called for an edge e of the restricted Delaunay triangulation $\mathcal{D}_{|L_j}(\mathcal{P})$ of some edge L_j in $\mathcal{L}$. The procedure inserts in $\mathcal{P}$ the center c_e of the restricted Delaunay ball $B(c_e, r_e)$ of e with largest ratio r_e.

refine-facet-or-edge. The procedure `refine-facet-or-edge`(f) is called for a facet f of the restricted Delaunay triangulation $\mathcal{D}_{|S_k}(\mathcal{P})$ of some facet S_k in $\mathcal{S}$. The procedure considers the center c_f of the restricted Delaunay ball $B(c_f, r_f)$ of f with largest radius r_f and performs the following:
- if c_f encroaches some edge e in $\cup_{L_j \in \mathcal{L}} \mathcal{D}_{|L_j}(\mathcal{P})$, call `refine-edge`$(e)$,
- else add c_f in $\mathcal{P}$.

refine-tet-facet-or-edge. The procedure `refine-tet-facet-or-edge`(t) is called for a cell t of $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$. It considers the circumcenter c_t of t and performs the following:
- if c_t encroaches some edge e in $\cup_{L_j \in \mathcal{L}} \mathcal{D}_{|L_j}(\mathcal{P})$, call `refine-edge`$(e)$,
- else if c_t encroaches some facet f in $\cup_{S_k \in \mathcal{L}} \mathcal{D}_{|S_k}(\mathcal{P})$,
  call `refine-facet-or-edge`(f),
- else add c_t in $\mathcal{P}$.

4 Output Mesh

At the end of the refinement process, the tetrahedra in $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ form the final mesh and the features of $\mathcal{L}$ and $\mathcal{S}$ are approximated by their respective restricted Delaunay triangulations. In this section we assume that the refinement process terminates, and we prove that after termination, each connected component O_l of the domain O is represented by a submesh formed with well sized and well shaped tetrahedra and that the boundary of this submesh is an accurate and homeomorphic approximation of $\mathrm{bd}\, O_l$.

Theorem 4.1. *If the meshing algorithm terminates, the output mesh $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ has the following properties.*

Size and shape. *The tetrahedra in $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ conform to the input sizing field and are well shaped (meaning that their radius-edge ratio is bounded by β_3).*

Homeomorphism. *There is an homeomorphism between $\mathcal{O}$ and $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ such that each face F of $\mathcal{F}$ is mapped to its restricted Delaunay triangulation $\mathcal{D}_{|F}(\mathcal{P})$.*

Hausdorff distance. *For each face F in $\mathcal{F}$, the Hausdorff distance between the restricted Delaunay triangulation $\mathcal{D}_{|F}(\mathcal{P})$ and F is bounded.*

Proof. The first point is a direct consequence of rules R5.2 and R5.1. The rest of this section is devoted respectively to the proof of the homeomorphism properties and to the proof of Hausdorff distance. □

The extended closed ball property.

To prove the homeomorphism between $\mathcal{O}$ and $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ we make use of the Edelsbrunner and Shah theorem [ES97]. In fact, because neither the domain $\mathcal{O}$, nor the union $\bigcup \mathcal{F}_2$ of faces with dimension at most 2 are assumed to be a manifold topological spaces, we make use of the version of Edelsbrunner and Shah theorem for non manifold topological spaces. This theorem is based on an extended version of the closed balled property recalled here for completeness.

Definition 3 (Extended closed ball property)**.** *A CW complex is a regular complex whose faces are topological balls. A set of point $\mathcal{P}$ is said to have the extended closed ball property with respect to a topological space $\mathcal{X}$ of $\mathbb{R}^d$ if there is a CW complex $\mathcal{R}$ with $\mathcal{X} = \bigcup \mathcal{R}$ and such that, for any subset $\mathcal{T} \subseteq \mathcal{P}$ whose Voronoi face $V(\mathcal{T}) = \bigcap_{p\in\mathcal{T}} V(p)$ has a non empty intersection with $\mathcal{X}$, the following holds.*

1. *There is a CW subcomplex $\mathcal{R}_\mathcal{T} \subset \mathcal{R}$ such that $\bigcup \mathcal{R}_\mathcal{T} = V(\mathcal{T}) \cap \mathcal{X}$.*
2. *Let $\mathcal{R}^0_\mathcal{T}$ be the subset of faces G in $\mathcal{R}_\mathcal{T}$ such that the interior of G is included in the interior of $V(\mathcal{T})$. There is a unique face $G_\mathcal{T}$ of $\mathcal{R}_\mathcal{T}$ which is included in all the faces of $\mathcal{R}^0_\mathcal{T}$.*
3. *If $G_\mathcal{T}$ has dimension j, $G_\mathcal{T} \cap \mathrm{bd}\, V(\mathcal{T})$ is a $j-1$-sphere.*
4. *For each face $G \in \mathcal{R}^0_\mathcal{T} \setminus \{G_\mathcal{T}\}$ with dimension k, $G \cap \mathrm{bd}\, V(\mathcal{T})$ is a $k-1$ ball.*

Furthermore, $\mathcal{P}$ is said to have the extended generic intersection property for $\mathcal{X}$ if for every subset $\mathcal{T} \subseteq \mathcal{P}$ and every face $G' \in \mathcal{R}_\mathcal{T} \setminus \mathcal{R}^0_\mathcal{T}$ there is a face $G \in \mathcal{R}^0_\mathcal{T}$ such that $G' \subseteq G$.

Theorem 4.2 ([ES97])**.** *If $\mathcal{X}$ is a topological space and $\mathcal{P}$ is a finite point set that has, with respect to $\mathcal{X}$, the extended closed ball property and the extended generic intersection property, $\mathcal{X}$ and $\mathcal{D}_{|\mathcal{X}}(\mathcal{P})$ are homeomorphic.*

In the following, we consider the final sampling $\mathcal{P}$ produced by the meshing algorithm and we show that $\mathcal{P}$ has the extended closed ball property and extended generic intersection property with respect to $\mathcal{O}$. For this, we need a CW complex $\mathcal{R}$ whose domain coincides with $\mathcal{O}$. We define $\mathcal{R}$ as $\mathcal{R} = \{V(\mathcal{T}) \cap F : \mathcal{T} \subseteq \mathcal{P}, F \in \mathcal{F}\}$, and our first goal is therefore to prove that each face in this complex is a topological ball.

Surface sampling

Let us first recall a few basic lemmas from the recently developed surface sampling theory [ACDL02, BO05]. They are hereafter adapted to our setting where the faces $S_k \in \mathcal{S}$ are patches of smooth closed surfaces.

Lemma 4.3 (Topological lemma). *[AB99] For any point $x \in \bigcup \mathcal{F}_2$, any ball $B(x, r)$ centered on x and with radius $r \leq \mathrm{lfs}(x)$ intersects any face F of $\mathcal{F}_2$ including x according to a topological ball.*

Lemma 4.4 (Long distance lemma). *[Dey06] Let x be a point in a face S_k of $\mathcal{S}$. If a line l through x makes a small angle $(l, l(x)) \leq \eta$ with the line $l(x)$ normal to S_k at x and intersects S_k in some other other point y, $\mathrm{d}(x, y) \geq 2\mathrm{lfs}(x)\cos(\eta)$.*

Lemma 4.5 (Chord angle lemma). *[AB99] For any two points x and y of S_k with $\mathrm{d}(x, y) \leq \eta\,\mathrm{lfs}(x)$ and $\eta \leq 2$, the angle between xy and T_x, the tangent plane of S_k at x, is at most* $\arcsin\frac{\eta}{2}$.

Lemma 4.6 (Normal variation lemma). *[AB99] Let x and y be two points of S_k with $\mathrm{d}(x, y) \leq \eta\min(\mathrm{lfs}(x), \mathrm{lfs}(y))$, $\eta \leq 1/3$. Let $n(x)$ and $n(y)$ be the normal vectors to S_k at x and y respectively. Assumed that $n(x)$ and $n(y)$ are oriented consistently, for instance toward the exterior of the smooth closed surface including S_k. Then the angle $(n(x), n(y))$ is at most $\frac{\eta}{1-3\eta}$.*

Lemma 4.7 (Facet normal lemma). *[AB99] Let pqr be a triangle of $\mathcal{D}_{|S_k}$ with a restricted surface Delaunay ball $B(c, \rho)$ such that $\rho \leq \eta\,\mathrm{lfs}(c)$. If p is the vertex with the largest angle of triangle pqr, the line l_f normal to triangle pqr and the lines $l(p), l(q), l(r)$ normals to S_k in p, q, r respectively, are such that $(l_f, l(p)) \leq \arcsin(\frac{\sqrt{3}\eta}{1-\eta})$ and $(l_f, l(q))$ or $l_f, l(r)) \leq \arcsin(\frac{\sqrt{3}\eta}{1-\eta}) + \frac{2\eta}{1-7\eta}$.*

One of the main notion in the theory of surface sampling is the notion of ε-sample, introduced by Amenta and Bern [AB99]. A sample $\mathcal{P}$ on a smooth surface S is an ε-sample if any point x of S is at distance at most $\varepsilon\mathrm{lfs}(x)$ from a point in $\mathcal{P}$. The notion of ε-sample can be extended to the surface patches S_k, and we also get the following lemma as an extension of the corresponding lemma for surfaces.

Lemma 4.8 (ε-sample lemma). *If $\mathcal{P}$ is an ε-sample of the surface patch S_k, for any point p of $\mathcal{P}$, the intersection $V(p) \cap S_k$ of its Voronoi cell $V(p)$ with S_k is included in the ball $B(p, \frac{\varepsilon}{1-\varepsilon}\mathrm{lfs}(p))$ with center p and radius $\frac{\varepsilon}{1-\varepsilon}\mathrm{lfs}(p)$.*

At the end of the algorithm, the set of sample point $\mathcal{P}$ is such that for any patch S_k of $\mathcal{S}$, the subset $\mathcal{P} \cap S_k$ is a loose α_2-sample of S_k. This means that any restricted Delaunay ball $B(c, r)$ circumscribed to a face in the restricted Delaunay triangulation $\mathcal{D}_{|S_k}(\mathcal{P} \cap S_k)$ has its radius r bounded by $\alpha_2\mathrm{lfs}(c)$. Boissonnat and Oudot [BO05, BO06] study loose $\varepsilon-$sample of smooth surfaces and one of their main result is the fact that any loose ε-sample of a closed smooth surface S is an ε'-sample of S with $\varepsilon' = \varepsilon(1 + O(\varepsilon)$. Due to our definition of the local feature size lfs for surface patches, the proof of this result can be adapted to the surface patches S_k that are part of smooth surfaces. The same adaptation works also for several instrumental lemmas that they use to prove the main result. The following lemma are the adaptation to surface patches of three lemmas given in [Boi06].

Lemma 4.9 (Projection lemma)**.** *[Boi06] Let $\mathcal{P}_k$ be an ε-sample of the smooth surface patch S_k for $\varepsilon < 0.24$. Any pair f and f' of two facets of $\mathcal{D}_{|S_k}(\mathcal{P}_k)$ sharing a common vertex p, have non overlapping orthogonal projections onto the tangent plane at p i.e. the projections of the relative interiors of f and f' do not intersect.*

Lemma 4.10 (Loose $\varepsilon-$sample lemma)**.** *[BO05, Boi06] Any loose* $\varepsilon-$sample *of the smooth surface patch S_k is ε'-sample of S with $\varepsilon' = \varepsilon(1 + O(\varepsilon))$.*

Lemmas 4.9 and 4.10 still hold if the function lfs is replaced, both in the definition of loose $\varepsilon-$sample and in the description of the small cylinder, by any Lipschitz sizing field σ such that $\sigma(x) \leq \mathrm{lfs}(x)$.

Proof of the homeomorphism properties

The following lemmas are related to the final sampling $\mathcal{P}$ produced by the algorithm. They assume that the sizing field $\sigma(x)$ is less than $\mathrm{lfs}(x)$ for any point $x \in \mathcal{F}$ and that the constant α_1 and α_2 used in the algorithm are small enough.

Lemma 4.11 (**Curve segments lemma**)**.** *Let $\mathcal{P}$ be the final sampling produced by the mesh generation algorithm of Section 3 and let L_j be a curve segment in $\mathcal{L}$.*

a) *Let $V(p, q)$ be a facet of $\mathcal{V}(\mathcal{P})$. The intersection $V(p, q) \cap L_j$ is non empty if and only if p and q are consecutive vertices on L_j. Consequently, a facet $V(p, q)$ of $\mathcal{V}(\mathcal{P})$ intersects at most one curve segment of $\mathcal{L}$.*

   If non empty, the intersection $V(p, q) \cap L_j$ is a single point, i.e. a 0-dimensional topological ball.

b) *Let $V(p)$ be a cell of $\mathcal{V}(\mathcal{P})$. The intersection $V(p) \cap L_j$ is empty if $p \notin L_j$ and a single curve segment, i.e. a 1-dimensional topological ball, otherwise.*

Proof. **Proof of proposition 4.11*a***. If the facet $V(p, q)$ intersects the curve segment L_j of $\mathcal{L}$, pq is an edge of $\mathcal{D}_{|L_j}(\mathcal{P})$, and Rule R1 implies that p and q both belong to L_j. Let us furthermore show that p and q are consecutive on L_j. We consider a restricted Delaunay ball $B(c_e, r_e)$ circumscribed to the edge $e = pq$. From Rule R2, $B(c_e, r_e)$ has a radius smallest than $\mathrm{lfs}(c_e)$ and therefore intersects L_j according to a topological ball (lemma 4.3). This topological ball is just the portion of $L_j(p, q)$ of L_j joining p to q. Therefore $L_j(p, q)$ is included in $B(c_e, r_e)$ which encloses no vertex of $\mathcal{P}$, and p and q are consecutive on L_j.

Conversely, let p and q be two points of $\mathcal{P}$ consecutive on L_j. From what precedes, the portion $L_j(p, q)$ of L_j joining p to q is not allowed to intersect a Voronoi facet except $V(p, q)$. Therefore $L_j(p, q)$ is included in $V(p) \cup V(q)$ and has to intersect $V(p, q)$.

ector of p and q and let $B(c, r)$

Let us show that, if non empty the intersection $V(p, q) \cap L_j$ is a single point. if $V(p, q)$ intersect L_j, then only the portion $L_j(p, q)$ of L_j joining p to q may intersect $V(p, q)$ and $L_j(p, q)$ is included in a restricted Delaunay ball $B(c, r)$ (circumscribed to pq) which by rule R2 has a radius less than $\alpha_1 \mathrm{lfs}(c)$. More than one intersection point between $L_j(p, q)$ and $V(p, q)$ implies that the curvature radius of L_j at some point of $L_j(p, q)$ is less than $\alpha_1 \mathrm{lfs}(c)$. However, owing to the Lipschitz property of lfs, for any point $x \in L_j(p, q)$, $\mathrm{lfs}(x) \geq \mathrm{lfs}(c)(1 - \alpha_1)$ and there is a contradiction if $\alpha_1 \leq 1/2$.

Proof of Proposition 4.11*b*. Each L_j intersects at list the Voronoi cells of its vertices and therefore cannot be included in a single cell. Therefore if some edge L_j intersects a Voronoi cell $V(p)$, it has to intersect some boundary facet $V(p, q)$ of $V(p)$ and it then results from

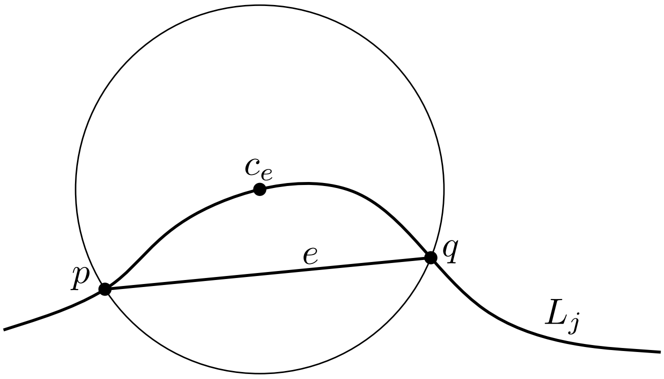

Figure 1: Intersection of a restricted Delaunay ball with L_j.

Proposition 4.11a that $p \in L_j$. Therefore any edge L_j intersects only the Voronoi cells of vertices lying on L_j.

If $p \in L_j$, Proposition 4.11 a implies that $\text{bd}\, V(p) \cap L_j$ is either a single point if p is a vertex of L_j, or two points if p is in the interior of L_j. This implies that $V(p) \cap L_j$ is a connected curve segment, *i.e.* a 1-dimensional topological ball. □

Lemma 4.12 (Surface patches lemma)**.** *Let $\mathcal{P}$ be the final sampling produced by the mesh generation algorithm of Section 3 and let S_k be a surface patch in $\mathcal{S}$.*

a) *Let $V(p,q,r)$ be an edge of $\mathcal{V}(\mathcal{P})$. If $V(p,q,r)$ intersects S_k, the vertices p, q and r belong to S_k. An edge of $\mathcal{V}(\mathcal{P})$ intersect at most one surface patch of $\mathcal{S}$. The intersection $V(p,q,r) \cap S_k$ is either empty or a single point, i.e. a 0-dimensional topological ball.*

b) *Let $V(p,q)$ be a facet of $\mathcal{V}(\mathcal{P})$. If $V(p,q)$ intersects S_k, the vertices p and q belong to S_k. If non empty, the intersection $V(p,q) \cap S_k$ is a single curve segment, i.e. a 1-dimensional topological ball .*

c) *Let $V(p)$ be a cell of $\mathcal{V}(\mathcal{P})$. The intersection $V(p) \cap S_k$ is empty if $p \notin S_k$ and a topological disk otherwise.*

Proof. **Proof of Proposition 4.12*a*.** If edge $V(p,q,r)$ intersects the patch S_k, the triangle pqr is a facet f of $\mathcal{D}_{|S_k}$ and Rule R3 implies that the vertices p, q and r belong to S_k. Let's assume for contradiction that edge $V(p,q,r)$ intersects more than one facet of $\mathcal{S}$ and let us consider two intersection points c_j and c_k consecutive on $V(p,q,r)$ such that c_j and c_k belong to different surface patches, S_j and S_k respectively. Rule R3 implies that the vertices p, q and r belong to both S_j and S_k, which therefore have to be adjacent facets in $\mathcal{F}_2$. Assume wlog that p is the vertex with largest angle of triangle $f = pqr$. We note $l_j(p)$ and $l_j(c_j)$ the lines normal to S_j at p and c_j respectively, and $l_k(p)$ and $l_k(c_k)$ the lines normal to S_k at p and c_k respectively, and l_f the line normal to the triangle $f = pqr$. Rule R4.1 implies that the circumradius of f is at most $\alpha_2 \min(\text{lfs}(c_j), \text{lfs}(c_k))$, and then, from facet normal lemma 4.7, we know that angles $(l_f, l_j(p))$ and $(l_f, l_k(p))$ are at most $\arcsin(\frac{\sqrt{3}\alpha_2}{1-\alpha_2})$. Therefore angle $(l_j(p), l_k(p))$ is $O(\alpha_2)$. Let us consider the normal vectors $n_j(p)$, normal to S_j at p and $n_k(p)$, normal to S_k at p, consistently oriented toward the exterior of the cell of $\mathcal{O}$ incident to S_j and S_k. The bound $O(\alpha_2)$ on angle $(l_j(p), l_k(p))$ imply that angle $(n_j(p), n_k(p))$ is either $\pi - O(\alpha_2)$ or $O(\alpha_2)$. The first case contradicts the angular condition, so assume that the second case occurs. Let $n_j(c_j)$ and $n_k(c_k)$ be the normal vectors to S_j at c_j and to S_k at c_k respectivement, still consistently oriented. We have

$$(n_j(c_j), n_k(c_k)) \leq (n_j(c_j), n_j(p)) + (n_j(p), n_k(p)) + (n_k(p), n_k(c_k)).$$

Rule R4.1 implies that $\mathrm{d}(c_j, p) \leq \alpha_2 \mathrm{lfs}(c_j)$ and $\mathrm{d}(c_k, p) \leq \alpha_2 \mathrm{lfs}(c_k)$, and then the normal variation lemma 4.6 implies that angles $(n_j(p), n_j(c_j))$ and $(n_k(p), n_k(c_k))$ are $O(\alpha_2)$. Thus, if $(n_j(p), n_k(p))$ is $O(\alpha_2)$, $(n_j(c_j), n_k(c_k))$ is also $O(\alpha_2)$, and this contradicts the fact that c_k and c_j are intersection points between $V(p, q, r)$ and the boundary of a cell in $\mathcal{O}$ that are consecutive on $V(p, q, r)$.

Let us now show that, if non empty, the intersection $V(p, q, r) \cap S_k$ is a single point. Let's assume for contradiction that $V(p, q, r)$ intersects S_k on more than one point and let c and c' be two intersection points, consecutive on the edge $V(p, q, r)$. Let $n(c)$, $n(c')$ and $n(p)$ be normal vectors of S_k at c, c' and p respectively, and assume that those vectors are consistently oriented, for instance towards the exterior of one of the cell bounded by S_k. We have:

$$(n(c), n(c')) \leq (n(c), n(p)) + (n(p), n(c')).$$

Rule R4.1 and facet normal lemma 4.7 imply that angles $(n(c), n(p))$ and $(n(c'), n(p))$ are $O(\alpha_2)$. Therefore angle $(n(c), n(c')$ is also $O(\alpha_2)$ which contradicts the fact that c and c' are consecutive intersections points along $V(p, q, r)$.

Proof of Proposition 4.12*b*.

We first prove that $V(p, q) \cap S_k$ includes no closed curve. If either p or q belong to S_k we adapt a proof given in [Dey06]. Assume for contradiction that $p \in S_k$ and that $V(p, q) \cap S_k$ includes a closed curve γ. Let x be a point on γ and let l be the line that lies the hyperplane h including $V(p, q)$ and is normal to γ at x. Line l is the projection on h of the normal to S_k at x. Therefore the direction $n(x)$ normal to S_k at x is such that $(n(x), l) \leq (n(x), l')$ for any other line l' in h. Let l' be the line trough x and parallel to the projection of $n(p)$ on h. Because $\mathcal{P}$ is a loose α_2-sampling of S_k and hence a $O(\alpha_2)$-sampling (lemma 4.10), $d(x, p) = O(\alpha_2)\mathrm{lfs}(p)$ and $d(p, q) \leq 2d(x, p) = O(\alpha_2)\mathrm{lfs}(p)$. Therefore, from the normal variation lemma 4.6, $(n(p), n(x))$ is $O(\alpha_2)$, and from the chord lemma 4.5, $(n(p), pq)$ is at least $\frac{\pi}{2} - O(\alpha_2)$, and

$$(n(p), l') = \frac{\pi}{2} - (n(p), pq) = O(\alpha_2).$$

Therefore,

$$(n(x), l) \leq (n(x), l') \leq (n(x), n(p)) + (n(p), l') = O(\alpha_2).$$

Line l has to intersect the closed curve γ in at least a second point y distinct from x. Then the contradiction comes from the long distance lemma 4.4 which says that $\mathrm{d}(x, y)$ should be at least $\mathrm{lfs}(p)(1 - O(\alpha_2))$ and from the ε-sample lemma 4.8 and loose $\varepsilon-$sample lemma 4.10 which imply that $\mathrm{d}(x, y)$ is $O(\alpha_2 \mathrm{lfs}(p))$. This proves that $V(p, q) \cap S_k$ includes no closed curve if either p or q or both belong to S_k.

Next we show $V(p, q) \cap S_k$ includes no curve if neither p nor q belong to S_k. The Voronoi cells partition S_k, let us consider the trace of $\mathcal{V}(\mathcal{P})$ on S_k. The boundaries of $V(p) \cap S_k$ appear on S_k as a set of cycles. We make a distinction between cycles of the first type that have vertices which are insersection of S_k with Voronoi edges and cycles of the second type that have no vertices and correspond to closed curves in the intersection of S_k with some Voronoi facets. For a sample point $p \in S_k$, the boundary $V(p) \cap S_k$ includes only first type cycles. Because any patch S_k includes at least three sample points yielding first type cycles, there are first type cycles. The second type cycles are disjoint but may be nested. We call outer a cycle of the second type that is not immediatly nested in another cycle of the second type. Observe that if there are cycles of the second type on S_k, there are necessarily outer cycles. Let γ be an outer cycle. The cycle γ is the trace of a closed curve in some $V(pq) \cap S_k$, and because it is an outer cycle it is surrounded by a cycle γ' of first type. But then the portion of S_k bounded by γ and γ' belong to a cell $V(p)$ and because γ' has vertices, p belongs to some triangle of $\mathcal{D}_{|S_k}$ and therefore to S_k which contradicts the existence of the second type cycle γ in $V(pq) \cap S_k$. Thus there is no second type cycle on S_k, and therefore no closed curve in the intersection of S_k with a Voronoi facet.

Note that, owing to Rule R3, if p or q do not belong to S_k, edge pq do not belong to a triangle of $\mathcal{D}_{|S_k}$, and therefore $V(pq) \cap S_k$ do not intersect any Voronoi edges. Thus, having proved that there is no closed curve in $V(pq) \cap S_k$, we have in fact proved that $V(pq) \cap S_k$ is empty if either p or q do not belong to S_k. (Note also that because there is no closed curve in the intersection of S_k with a Voronoi facet, there is no edge in the restricted triangulation $\mathcal{D}_{|S_k}$ that does not belong to a triangle in $\mathcal{D}_{|S_k}$.)

It remains to show that, if non empty, the intersection $V(pq) \cap S_k$ is a topological ball. Because it includes no closed curve, such an intersection is a set of disjoint curve segments, with two boundary points each, and $bd(V(pq) \cap S_k)$ is a set of points with even cardinality. Then, we note that $\mathrm{bd}\,(V(p,q) \cap S_k) = (\mathrm{bd}\,V(p,q) \cap S_k) \cup (V(p,q) \cap \mathrm{bd}\,S_k)$. From Proposition 4.12$a$, each Voronoi edge in $\mathrm{bd}\,V(p,q)$ intersects S_k in at most a single point and such an intersection point corresponds to a facet incident to the edge pq in $\mathcal{D}_{|S_k}(\mathcal{P})$. The projection lemma 4.9 implies that there are at most two facets of $\mathcal{D}_{|S_k}(\mathcal{P})$ incident to a given edge and therefore $\mathrm{bd}\,V(p,q) \cap S_k$ includes at most two points. Lemma 4.11 implies that $V(p,q) \cap \mathrm{bd}\,S_k$ includes at most one point. Thus the number of intersection points in $\mathrm{bd}\,(V(p,q) \cap S_k)$ is at most three and because it has to be an even number, it is zero or two, which proves that the intersection $V(p,q) \cap S_k$ is either empty or a single curve segment, *i.e.* a topological ball.

Proof of Proposition 4.12*c*.

Let us first show that if $V(p) \cap S_k$ is not empty, $p \in S_k$. The patch S_k includes at least three vertices and therefore may not be included in the interior of $V(p)$. Therefore, if $V(p) \cap S_k$ is not empty, S_k intersects at least one facet $V(p,q)$ on $\mathrm{bd}\,V(p)$. From lemma 4.12b, $V(p,q) \cap S_k$ is a topological segment, with, from lemma 4.11, at most one endpoint in $V(p,q) \cap \mathrm{bd}\,S_k$, and therefore at least one endpoint on an edge $V(p,q,r)$ of $V(p)$. Thus, the triangle pqr belongs to $\mathcal{D}_{|S_k}(\mathcal{P})$ and rule R3 implies that $p \in S_k$.

Then, if $p \in S_k$, we show that $V(p) \cap S_k$ is a two dimensional topological ball by proving that its boundary $\mathrm{bd}\,(V(p) \cap S_k)$ is a one dimensional topological sphere. Indeed, because of the ε-sample lemma 4.8, loose $\varepsilon-$sample lemma 4.10 and of the normal variation lemma 4.6, the normal vector n_x to S_k at a point x of $V(p) \cap S_k$ is closed to the normal vector n_p of S_k at p. Therefore, $V(p) \cap S_k$ is a terrain above the tangent plane to S_k in p, and a 1-dimensional topological sphere on such a terrain bounds a 2-dimensional topological ball.

Assume first that p belongs to the interior of S_k. Then, from lemma 4.11, $V(p)$ intersects no edge of $\mathcal{L}$ and the intersection $V(p) \cap \mathrm{bd}\,S_k$ is empty. From lemma 4.12b, for each facet $V(p,q)$ on $\mathrm{bd}\,V(p)$ the intersection $V(p,q) \cap S_k$ is either empty or a 1-dimensional topological ball with two endpoints on edges of $V(p)$. Therefore, the intersection $\mathrm{bd}\,V(p) \cap S_k$ is a set of topological segments, that are intersection of the patch S_k with Voronoi facets and form cycles on $\mathrm{bd}\,V(p)$. Each such cycle correspond in the restricted Delaunay triangulation $\mathcal{D}_{|S_k}(\mathcal{P})$ to a cycle of adjacent triangles forming a topological ball around vertex p. Then, projection lemma 4.9 implies that there is at most one such cycle of adjacent triangles in $\mathcal{D}_{|S_k}(\mathcal{P})$, and therefore only one cycle in $\mathrm{bd}\,V(p) \cap S_k$. As a consequence, $\mathrm{bd}\,(V(p) \cap S_k) = \mathrm{bd}\,V(p) \cap S_k$ is a 1-dimensional topological sphere.

If p belongs to the interior of an edge L_j of $\mathrm{bd}\,S_k$, we know from lemma 4.11b that the cell $V(p)$ intersect L_j but no other curve segment in $\mathcal{L}$ so that $V(p) \cap \mathrm{bd}\,S_k$ reduces to $V(p) \cap L_j$ which is a 1-dimensional topological ball. Each of the two Voronoi facets on $\mathrm{bd}\,V(p)$ intersecting

L_j, intersects S_k according to a topological segment with one endpoint on $\mathrm{bd}\, V(p) \cap L_j$ and one endpoint on an edge of $V(p)$. Any other non empty intersection $V(p,q) \cap S_k$ of S_k with a facet of $V(p)$ is a 1-dimensional topological ball with two endpoints on edges of $V(p)$. The non empty intersections $V(p,q) \cap S_k$ form a chain of curve segments ending on the two points $\mathrm{bd}\, V(p) \cap L_j$ plus may be possibly additionnal cycles of curve segments. Projection lemma 4.9 implies that $\mathrm{bd}\, V(p) \cap S_k$ reduces in fact to the chain joining the two points of $\mathrm{bd}\, V(p) \cap L_j$, and therefore $\mathrm{bd}\,(V(p) \cap \mathrm{bd}\, S_k) = (\mathrm{bd}\, V(p) \cap S_k) \cup (V(p) \cap \mathrm{bd}\, S_k)$ is a 1-dimensional topological sphere.

At last, if p is a vertex of S_k, p belongs to two edges L_i and L_j of $\mathrm{bd}\, S_k$ incident to p. In this case $V(p) \cap L_i$ (resp.$V(p) \cap L_j$) is a 1-dimensional topological sphere with endpoints in p and $\mathrm{bd}\, V(p) \cap L_i$ (resp. in p and $\mathrm{bd}\, V(p) \cap L_j$). Then $\mathrm{bd}\, V(p) \cap S_k$ is a set of 1-dimensional topological segments forming a chain joining $\mathrm{bd}\, V(p) \cap L_i$ and $\mathrm{bd}\, V(p) \cap L_i$ plus additionnal cycles. Projection lemma 4.9 implies that there is no cycle in $\mathrm{bd}\, V(p) \cap S_k$. Then, $\mathrm{bd}\,(V(p) \cap S_k)$ is a 1-dimensional topological sphere obtained as the concatenation of $V(p) \cap L_i$, $V(p) \cap L_j$ and the chain of non empty intersections $V(p,q) \cap S_k$ between S_k and the facets of $\mathrm{bd}\, V(p)$. □

Lemma 4.13 (Cells lemma). *Let $\mathcal{P}$ be the final sampling produced by the mesh generation algorithm of Section 3 and let C_l be a cell in $\mathcal{C}$.*

a) *Let $V(p)$ be a cell of $\mathcal{V}(\mathcal{P})$. The intersection $C_l \cap V(p)$ is non empty if and only if $p \in C_l$.*

b) *The intersection of C_l with any Voronoi face (edge, facet or cell) of $\mathcal{V}(\mathcal{P})$ is a topological ball*

Proof. **Proof of Proposition 4.13*a*.**

It results from lemmas 4.11b and 4.12c, that the cell $V(p)$ intersects the boundary $\mathrm{bd}\, C_l$ if and only iff p belongs to $\mathrm{bd}\, C_l$. Therefore if p belongs to the interior $\mathrm{int}\, C_l$ of C_l, $V(p)$ is included in $\mathrm{int}\, C_l$, and if p does not belong to C_l, $V(p)$ and C_l are disjoint.

Proof of Proposition 4.13*b*.

Let $V(p,q,r)$ be a Voronoi edge. Lemma 4.12c ensures that $V(p,q,r)$ intersect $\mathrm{bd}\, C_l$ in at most one point, therefore $V(p,q,r) \cap C_l$ is either empty or $V(p,q,r)$ or a subsegment of $V(p,q,r)$. Therefore, if non empty, $V(p,q,r) \cap C_l$ is a topological 1-dimensional ball.

Let $V(p,q)$ be a Voronoi facet. We consider the intersection $V(p,q) \cap \mathrm{bd}\, C_l$. From lemma 4.12$b$, there are different case according to the location of p and q on $\mathrm{bd}\, C_l$.

- If p and q do not belong to the same surface patch S_k $V(p,q) \cap \mathrm{bd}\, C_l$ is empty.
- If p and q belong to the same surface patch S_k incident to C_l but not the the same curve segment of $\mathcal{L}$. $V(p,q) \cap \mathrm{bd}\, C_l$ reduces to $V(p,q) \cap S_k$. It is a 1-dimensional topological ball whose boundary is included in $\mathrm{bd}\, V(p,q)$.
- If p and q belong to the same curve segment L_j on the boundary of C_l, $V(p,q) \cap \mathrm{bd}\, C_l$ is the union of the 1-dimensional topological balls $\{V(p,q) \cap S_k : S_k \in \mathrm{bd}\, C_l, L_j \subset S_k\}$. In this case, $V(p,q) \cap \mathrm{bd}\, C_l$ is the union of two 1-dimensional topological balls which share the endpoint $V(p,q) \cap L_j$ and each have another endpoint included in $\mathrm{bd}\, V(p,q)$, it is therefore a 1-dimensional topological balls with two boundary points on $\mathrm{bd}\, V(p,q)$.

The intersection $V(p,q) \cap C_l$ is either empty or equal to the 2-dimensional topological ball $V(p,q)$ or equal to a portion of $V(p,q)$ determined by $V(p,q) \cap \mathrm{bd}\, C_l$. The 1-dimensional topological ball $V(p,q) \cap \mathrm{bd}\, C_l$ whose boundary is a 0-dimensional sphere included in $\mathrm{bd}\, V(p,q)$ split $V(p,q)$ in two 2-dimensional topological balls and $V(p,q) \cap C_l$ is one of them.

Let $V(p)$ be a cell of $\mathcal{V}(\mathcal{P})$. If p does not belong to C_l, $V(p) \cap C_l$ is empty. If p belong to the interior of C_l $V(p)$ is included in C_l, and $V(p) \cap \mathrm{bd}\, C_l = V(p)$ is a 3-dimensional topological ball. In the other cases, p belongs to $\mathrm{bd}\, C_l$ and is either in the interior of a surface patch S_k on $\mathrm{bd}\, C_l$, or in the interior of a curve segment L_j on $\mathrm{bd}\, C_l$ or p is a vertex of C_l. In all three cases, $V(p) \cap \mathrm{bd}\, C_l$ is a union of the set of intersections $V(p) \cap S_k$ fo each facet S_k of $\mathrm{bd}\, C_l$

including p. Each $V(p) \cap S_k$ is a 2-dimensional topological ball (lemma 4.12c), and their union is a 2-dimensional topological ball whose boundary is a 1-dimensional sphere on $\mathrm{bd}\, V(p)$ (see the proof of lemma 4.12c) . Therefore, $V(p) \cap \mathrm{bd}\, C_l$ split $V(p)$ in two 3-dimensional topological balls and $V(p) \cap \mathrm{bd}\, C_l$ is one of them. □

4.0.3 Proof of the homeomorphism properties

It follows from the previous subsection that for every subset $\mathcal{T} \subseteq \mathcal{P}$ and every face $F \in \mathcal{F}$, the intersection $V(\mathcal{T}) \cap F$ is either empty or a topological ball. In the following, we assume that the sample set $\mathcal{P}$ has some genericity property with respect to $\mathcal{F}$. (This genericity property can be achieved by a small perturbation in which each vertex of $\mathcal{P}$ keeps the same set of incidences on faces of $\mathcal{F}$). Namely we assume that, if non empty, the Voronoi face $V(\mathcal{T})$ of a subset $\mathcal{T} \subset \mathcal{P}$ with cardinality k has dimension $k' = 4 - k$ and that, if non empty, its intersection $V(\mathcal{T}) \cap F$ with a j-dimensional face in $\mathcal{F}$ has dimension $k' + j - 3$.

Lemma 4.14. *If $\mathcal{P}$ is the final sampling produce by the algorithm of section 3, The set $\mathcal{R} = \{V(\mathcal{T}) \cap F : \mathcal{T} \subset \mathcal{P}, F \in \mathcal{F}\}$ form a CW complex and, under the genericity condition, $\mathcal{P}$ has the extended closed ball property and extended generic property for $\mathcal{O}$.*

Proof. Lemmas 4.11, 4.12, and 4.13 show that each element in $\mathcal{R}$ is a topological ball. Obviously, the faces in $\mathcal{R}$ have disjoint interior and the boundary of each face in $\mathcal{R}$ is a union of faces in $\mathcal{R}$. Therefore $\mathcal{R}$ is a CW complex. For any subset $\mathcal{T} \in \mathcal{P}$, we set $\mathcal{R}_{\mathcal{T}} = \{V(\mathcal{T}) \cap F : F \in \mathcal{F}\}$.

We show that that the genericity assumption implies the extended generic property. Indeed it implies that, if $V(\mathcal{T}) \cap F \neq \emptyset$, $\mathrm{int}\,(V(\mathcal{T}) \cap F) = \mathrm{int}\, V(T) \cap \mathrm{int}\, F$, which yields that for any subset $\mathcal{T} \subset \mathcal{P}$, $\mathcal{R}^0_{\mathcal{T}} = \mathcal{R}_{\mathcal{T}}$.

In [CDR07], Cheng, Dey and Ramos, show that the conditions 1-4 of the extended closed ball property are satisfied by $\mathcal{P}$ if the following two properties are satisfied.

- **P1:** If $V(\mathcal{T})$ is a k-dimensional Voronoi face of $\mathcal{V}(\mathcal{P})$ and F a j-dimensional face of $\mathcal{F}$, the intersection $V(\mathcal{T}) \cap F$ is either empty or a $k + j - 3$-dimensional topological ball.
- **P2:** For any subset $\mathcal{T} \subset \mathcal{P}$, there is a unique lowest dimensional element $F_{\mathcal{T}} \in \mathcal{F}$ such that $F_{\mathcal{T}}$ intersects $V(T)$ and $F_{\mathcal{T}}$ is included in and all the faces in $\mathcal{F}$ intersecting $V(\mathcal{T})$.

Property P1 is granted by the genericity condition and by Lemmas 4.11, 4.12, and 4.13. Let us show that Lemmas 4.11,4.12, and 4.13 also yield Property P2. Let $V(p, q, r)$ be an edge of $\mathcal{V}(\mathcal{P})$. Either $V(p, q, r)$ is included in a cell of $\mathcal{F}$, or $V(p, q, r)$ intersects a single surface patch S_k and the cells incident to S_k. Let $V(p, q)$ be a facet of $\mathcal{V}(\mathcal{P})$. Either $V(p, q)$ is included in a cell of $\mathcal{F}$, or it intersect a single surface patch S_k and the incident cells or it intersects a single curve segment L_j and all the surface patches and cells including L_i. (See Figure 2.) Let $V(p)$ be a cell of $\mathcal{V}(\mathcal{P})$. Either $V(p)$ is included in a cell of $\mathcal{F}$, it intersects a single surface patch S_k if $p \in \mathrm{int}\, S_k$, or it intersects a single curve segment L_j if $p \in \mathrm{int}\, L_j$ or p is a vertex of $\mathcal{F}$.

For completeness, we recall here the argument of Cheng, Dey and Ramos to show that properties P1 and P2 ensures the condition 1-4 of the extended closed ball property.

Condition 1 results from P1, and condition 2 results from P2 with, for any Voronoi face $V(\mathcal{T})$, $G_{\mathcal{T}} = F_{\mathcal{T}} \cap V(\mathcal{T})$.

To prove Condition 3, we first notice that P2 implies that $\mathrm{bd}\, F_{\mathcal{T}} \cap V(\mathcal{T})$ is empty. Indeed, if this intersection was not empty, some face in $\mathrm{bd}\, F_{\mathcal{T}}$ would intersect $V(\mathcal{T})$ and $F_{\mathcal{T}}$ would not be the lowest dimensional face of $\mathcal{F}$ intersecting $V(\mathcal{T})$. Then, $G_{\mathcal{T}} \cap \mathrm{bd}\, V(\mathcal{T}) = F_{\mathcal{T}} \cap \mathrm{bd}\, V(\mathcal{T})$ is just $\mathrm{bd}\,(F_{\mathcal{T}} \cap V(\mathcal{T}))$ which is a topological sphere from P1.

At last, for any face $F \in \mathcal{F}$ that is not $F_{\mathcal{T}}$ and intersects $V(\mathcal{T})$, $G = F \cap V(\mathcal{T})$ is such that $G \cap \mathrm{bd}\, V(\mathcal{T}) = F \cap \mathrm{bd}\, V(\mathcal{T})$. Property P1 implies that $\mathrm{bd}\,(F \cap V(\mathcal{T}))$ is a topological sphere of

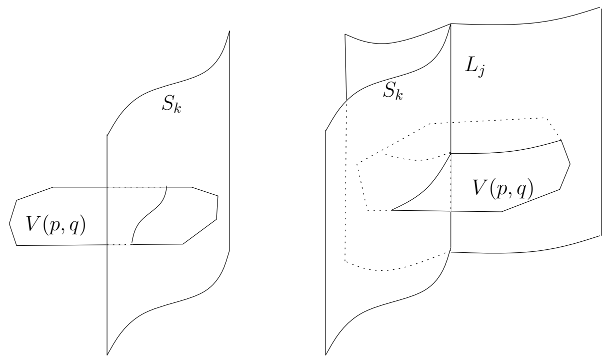

Figure 2: Intersection with a Voronoi facet $V(p,q)$. Left $V(p,q)$ intersects no curved segment L_j. Right $V(p,q)$ intersects a single curved segment L_j.

dimension $l-1$ if the dimension of $F \cap V(\mathcal{T})$ is l. Because it contains $F_{\mathcal{T}}$, the boundary $\mathrm{bd}\, F$ intersects $V(T)$ and $\mathrm{bd}\, F \cap V(\mathcal{T}))$ is a $l-1$-topological ball, which implies that $F \cap \mathrm{bd}\, V(T)$ is a $l-1$-topological ball. □

As a conclusion, under the genericity hypothesis and also provided that the constant α_1 and α_2 used by the algorithm have small enough values (as required by the sampling lemmas 4.3-4.10) the final sample set $\mathcal{P}$ has the extended closed ball property and extented generic intersection property for $\mathcal{O}$. Therefore we can conclude by theorem 4.2 that $\mathcal{O}$ and $\mathcal{D}_{|\mathcal{O}}(\mathcal{P})$ are homeomorphic. Moreover, because the proof of theorem 4.2 constructs the isomorphism step by step between each face $V(\mathcal{T}) \cap F \in \mathcal{R}$ and the corresponding face in $\mathcal{D}_{|F}(\mathcal{P})$ in non decreasing order of dimension, the resulting isomorphism is such that each face F of $\mathcal{F}$ is mapped to its restricted Delaunay triangulation $\mathcal{D}_{|F}(\mathcal{P})$.

4.0.4 Hausdorff distance

We prove here that the meshing algorithm allows to control the Hausdorff distance between F and the approximating linear complex $\mathcal{D}_{|F}(\mathcal{P})$ through the sizing field σ.

Let us first consider a curve segment L_j in $\mathcal{L}$. For each edge $e = pq$ in $\mathcal{D}_{|L_j}(\mathcal{P})$, both edge e and the portion $L_j(p,q)$ of L_j joining p to q are included in the restricted Delaunay $B(c_e, r_e)$ circumscribed to e. The Hausdorff distance between $L_j(p,q)$ and e is therefore less than r_e which, from rule R2, is less $\alpha_1 \sigma(c_e)$ and the Hausdorff distance between L_j and $\mathcal{D}_{|L_j}(\mathcal{P})$ is less than $\alpha_1 \max_{x \in L_j} \sigma(x)$.

Let us then consider a surface patch S_k. Each triangle pqr in $\mathcal{D}_{|S_k}(\mathcal{P})$ is included in its restricted Delaunay ball $B(c,r)$ with radius $r \leq \alpha_2 \sigma(c)$ and therefore each point of pqr is at distance less than $\alpha_2 \sigma(c)$ from S_k. From rule R4.1 and ε-sample lemma 4.8 and loose $\varepsilon-$sample lemma 4.10, we know that each point x in S_k is at distance $O(\alpha_2)\sigma(p)$ from its closest sample point p. The Hausdorff distance between S_k and $\mathcal{D}_{|S_k}(\mathcal{P})$ is less than $O(\alpha_2) \max_{x \in S_k} \sigma(x)$.

5 Termination

This section proves that the refinement algorithm in Section 3 terminates, provided that the constants α_1 and α_2, β_2 and β_3 involved in refinement rules are judiciously chosen. The proof of termination is, as usual, based on a volume argument. This requires to lower bound the distance between any two vertices inserted in the mesh.

For each vertex p in the mesh, we denote by $r(p)$ the insertion radius of p, that is the length of the shortest edge incident to vertex p right after the insertion of p. Recall that we denote by $\mathcal{F}_2$ the subset of faces in $\mathcal{F}$ with dimension at most 2, $\mathcal{F}_2 = \mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}$. For any point p, let $\mathcal{F}_2(p)$ the subset of faces in $\mathcal{F}_2$ including the point p, and let and $\delta(p)$ be the distance from p to $\mathcal{F}_2 \setminus \mathcal{F}_2(p)$.

The algorithm depends on four parameters α_1, α_2, β_2 and β_3, and on a sizing field σ. Constants α_1 and α_2 are assumed to be small enough and such that $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. The sizing field σ is assumed to be a Lipschitz function smaller than $\mathrm{lfs}(x)$ on $\mathcal{F}$. Let μ_0 be the maximum of σ over $\mathcal{F}_2$, and σ_0 the minimum of σ over $\mathcal{O}$.

Lemma 5.1. *For any point x in a face $F \in \mathcal{F}_2$, we have*

$$\delta(x) \geq \min\left(\mathrm{d}(x, \mathrm{bd}\, F), \mathrm{lfs}(x)\right).$$

Proof. We show that we have either $\delta(x) \geq \mathrm{d}(x, \mathrm{bd}\, F)$ or $\delta(x) \geq \mathrm{lfs}(x)$. Let $G \in \mathcal{F}_2$ be a face that do not contain x, and y a point of G.

- If x and y are not interrelated, then $\mathrm{d}(x, y) \geq \mathrm{lfs}(x)$.
- If x and y are interrelated, then F and G intersect, and there exists $w \in F \cap G$ with $\mathrm{d}(x, w) \leq \lambda_0$ and $\mathrm{d}(y, w) \leq \lambda_0$. In that case, inequality 1 of the angular hypothesis implies that $\mathrm{d}(x, y) \geq \mathrm{d}(x, F \cap G)$, and therefore $\mathrm{d}(x, y) \geq \mathrm{d}(x, \mathrm{bd}\, F)$.

In both case, $\mathrm{d}(x, y) \geq \mathrm{d}(x, \mathrm{bd}\, F)$ or $\mathrm{d}(x, y) \geq \mathrm{lfs}(x)$, hence the lemma. □

An easy consequence is that, if a point x belongs to the interior of some face $F \in \mathcal{F}_2$ and is at distance at least $\mathrm{d}(x, \mathrm{bd}\, F) \leq \sigma_0$ from the boundary of F then $\delta(x) \geq \mathrm{d}(x, \mathrm{bd}\, F)$.

Lemma 5.2. *For some suitable values of α_1,α_2, β_2 and β_3, there are constants η_2 and η_3 such that $\alpha_1 \leq \eta_3 \leq \eta_2 \leq 1$ and such that the following invariants are satisfied during the execution of the algorithm.*

$$\forall p \in \mathcal{P}, \quad r(p) \geq \alpha_1 \sigma_0 \tag{2}$$

$$\delta(p) \geq \begin{cases} \alpha_1 \sigma_0 & \textit{if } p \in \bigcup \mathcal{L} \\ \frac{\alpha_1 \sigma_0}{\eta_2} & \textit{if } p \in \bigcup \mathcal{F}_2 \setminus \bigcup \mathcal{L} \\ \frac{\alpha_1 \sigma_0}{\eta_3} & \textit{if } p \in \mathcal{O} \setminus \bigcup \mathcal{F}_2 \end{cases} \tag{3}$$

Proof. The proof is an induction. Invariants (2) and (3) are satisfied by the set $\mathcal{Q}$ and by the set $\mathcal{P}_0$ of initial vertices. We prove that invariants(2) and (3) are still satisfied after the application of any of the refinement rules R1-R5 if the following values are set:

$$\begin{array}{rclrcl} \alpha_1 & = & \frac{1}{(\sqrt{2}+2)\nu_0,(\nu_0+1)} & \alpha_2 & = & \frac{1}{\nu_0+1}, \\ \beta_2 & = & (\sqrt{2}+2)\nu_0, & \beta_3 & = & (\sqrt{2}+2)\nu_0\,(\nu_0+1)\,, \\ \frac{1}{\eta_2} & = & (\sqrt{2}+1)\nu_0, & \frac{1}{\eta_3} & = & (\sqrt{2}+2)\nu_0, \\ \text{where } \nu_0 & = & \frac{2\mu_0}{\sigma_0}. & & & \end{array} \tag{4}$$

Rule R1 When rule R1 applies, the new vertex c belongs to a curve segment L_j and is the center of a Delaunay ball circumscribed to an edge pq of $\mathcal{D}_{|L_j}$ such that at least one of its vertices, say q does not belong to L_j.

$$r(c) = \|cq\| \geq \mathrm{d}(q, L_j) \geq \alpha_1 \sigma_0$$

where last equation holds by recurrence hypothesis. So, invariant (2) is satisfied.

For the second part, we notice that the Delaunay ball $B(c, r(c))$ is empty of vertices and therefore

$$d(c, \mathcal{P}) \geq r(c) \geq \alpha_1 \sigma_0.$$

It means that c is at least at distance $\alpha_1\sigma_0$ from the extremities of L_j. Then lemma 5.1 shows that invariant (3) is satisfied.

Rule R2 When rule R2 applies, the new vertex c belongs to a curve segment L_j and is the center of a restricted Delaunay ball $B(c, r)$ circumscribing an edge e of $\mathcal{D}_{|L_j}(\mathcal{P})$ and such that $r \geq \alpha_1\sigma(c)$. The radius of insertion $r(c)$ of c is just r and satisfies

$$r(c) = r \geq \alpha_1 \sigma(c) \geq \alpha_1 \sigma_0.$$

Thus, invariant (2) is satisfied. Then, invariant (3) can be proved exactly as in the case of rule R1.

Rule R3 When rule R3 applies, the new vertex c_f belongs to a surface patch S_k and is the center of a Delaunay ball $B(c_f, r_f)$ circumscribing a facet f of $\mathcal{D}_{|S_k}$. At least one of the vertices of f vertices, say p, does not belong to S_k. The insertion radius of c_f is r_f and,

$$r_f = \|c_f p\| \geq \mathrm{d}(p, S_k).$$

By induction hypothesis, $\mathrm{d}(p, S_k)$ is at least $\alpha_1\sigma_0$, and invariant (2) is satisfied.

To prove the invariant (3), we bound the distance $\mathrm{d}(c_f, \mathrm{bd}\, S_k)$ and apply lemma 5.1. Let L_i be any 1-dimensional feature L_i bounding S_k. Let y be the point of L_i closest to c_f, and let q be the sample point in $\mathcal{P} \cap L_i$ closest to y. Then

$$\begin{aligned} \mathrm{d}(c_f, L_i) \quad &= \mathrm{d}(c_f, y) \quad \geq \mathrm{d}(c_f, q) - \mathrm{d}(y, q) && (5)\\ &\geq \mathrm{d}(c_f, \mathcal{P} \cap L_i) - \mathrm{d}(y, \mathcal{P} \cap L_i) && (6) \end{aligned}$$

Because the ball $B(c_f, r_f)$ is a Delaunay ball, $\mathrm{d}(c_f, \mathcal{P} \cap L_i) \geq r_f$. Rules R1 and R2 do no longer apply when rule R3 is applied. We know from the proof of lemma 4.11 that, at that time, L_i is covered by the union of restricted Delaunay balls centered on L_i. Therefore, there is a restricted Delaunay ball $B(c_{e_1}, r_{e_1})$, circumscribed to an edge e_1 of $\mathcal{D}_{|L_i}(\mathcal{P})$ and containing y. Let p_1 be one of the vertices of e_1. Then

$$\mathrm{d}(y, \mathcal{P} \cap L_i) \leq \mathrm{d}(y, p_1) \leq 2r_{e_1} \leq 2\alpha_1\mu_0$$

Hence,

$$\mathrm{d}(c_f, L_i) \geq r_f - 2\alpha_1\mu_0. \tag{7}$$

- If the vertex p does not belongs to $\mathcal{F}_2$, we have by induction hypothesis,

$$r_f = \mathrm{d}(c_f, p) \geq \frac{\alpha_1\sigma_0}{\eta_3},$$

hence

$$\mathrm{d}(c_f, L_i) \geq \frac{\alpha_1 \sigma_0}{\eta_3} - 2\alpha_1 \mu_0$$

Therefore $\mathrm{d}(c_f, \mathrm{bd}\, S_k) \geq \frac{\alpha_1 \sigma_0}{\eta_3} - 2\alpha_1 \mu_0$. Lemma 5.1 implies that $\delta(c_f) \geq \frac{\alpha_1 \sigma_0}{\eta_3} - 2\alpha_1 \mu_0$ and invariant (3), is satisfied if:

$$\left(\frac{1}{\eta_3} - \frac{1}{\eta_2}\right) \geq \frac{2\mu_0}{\sigma_0} \tag{8}$$

- If the vertex p lies on $\mathcal{F}_2$, but p and c_f are not interrelated, then

$$r_f = \mathrm{d}(c_f, p) \geq \mathrm{lfs}(c_f),$$

hence, as $\mathrm{lfs}(c_f) \geq \sigma_0$, we have

$$\mathrm{d}(c_f, L_i) \geq \sigma_0 - 2\alpha_1 \mu_0.$$

Therefore $\mathrm{d}(c_f, \mathrm{bd}\, S_k) \geq \sigma_0 - 2\alpha_1 \mu_0$. From Lemma 5.1 $\delta(c_f) \geq \sigma_0 - 2\alpha_1 \mu_0$ and invariant (3) is satisfied if:

$$\sigma_0 - 2\alpha_1 \mu_0 \geq \frac{\alpha_1 \sigma_0}{\eta_2}$$

id est:

$$\frac{1}{\alpha_1} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0}. \tag{9}$$

- If $p \in F_i \in \mathcal{F}_2$, and is interrelated with c_f, then there exists $w \in F_i \cap S_k$, so that

$$\begin{aligned} \mathrm{d}(p, w) &\leq \lambda_0 \\ \mathrm{d}(c_f, w) &\leq \lambda_0 \end{aligned}$$

Then, by angular hypothesis,

$$r_f^2 = \mathrm{d}(c_f, p)^2 \geq \mathrm{d}(c_f, S_k \cap F_i)^2 + \mathrm{d}(p, S_k \cap F_i)^2. \tag{10}$$

Thus $\mathrm{d}(c_f, S_k \cap F_i) \leq r_f$, which proves that sample point p cannot lie in a curve segment of $\mathcal{L}$. Indeed if F_i was some $L_i \in \mathcal{L}$, $S_k \cap F_i$ would be a vertex in $\mathcal{Q}$ included in $B(c_f, r_f)$ which contradicts the fact that $B(c_f, r_f)$ is a Delaunay ball. Therefore p belongs to the interior of some surface patch $S_i \in \mathcal{S}$ and the induction hypothesis implies that

$$\mathrm{d}(p, S_k \cap F_i) \geq \frac{\alpha_1 \sigma_0}{\eta_2},$$

and therefore

$$r_f^2 \geq \mathrm{d}(c_f, \mathrm{bd}\, S_k)^2 + \left(\frac{\alpha_1 \sigma_0}{\eta_2}\right)^2. \tag{11}$$

Because, $B(c_f, r_f)$ do not enclose any sample point, we have as in (7)

$$r_f \leq \mathrm{d}(c_f, \mathrm{bd}\, S_k) + 2\alpha_1 \mu_0. \tag{12}$$

Set for a while : $x = \mathrm{d}(c_f, \mathrm{bd}\, S_k)$, $a = 2\alpha_1\mu_0$ and $b = \frac{\alpha_1\sigma_0}{\eta_2}$. Then, equations (11) and (12) imply that:

$$(x+a)^2 \geq x^2 + b^2$$

i.e.:

$$x \geq \frac{b^2 - a^2}{2a}.$$

From Lemma 5.1 Invariant (3) is satisfied if $x \geq b$, that is, if:

$$b^2 - 2ab - b^2 \geq 0,$$

which is ensured if $b \geq a(1+\sqrt{2})$, or

$$\frac{1}{\eta_2} \geq 2(\sqrt{2}+1)\frac{\mu_0}{\sigma_0}. \tag{13}$$

To summarize, Invariant (3) is still verfied in this case if equation (13) holds.

Rule R4 When rule R4 is applied, the algorithm considers a point c_f on a surface patch S_k. Point c_f is the center of a restricted Delaunay ball $B(c_f, r_f)$ of a facet $f \in \mathcal{D}_{|S_k}(\mathcal{P})$ such that either:

$$r_f \geq \alpha_2\sigma(c_f)$$

or

$$r_f \geq \beta_2 l_{min}(f)$$

where $l_{min}(f)$ is the length of the smallest edge of f.

- In the first case, $r_f \geq \alpha_2\sigma(c) \geq \alpha_1\sigma_0$.
- In the second case, by induction hypothesis, $l_{min}(f) \geq \alpha_1\sigma_0$. Hence $r_f \geq \beta_2\alpha_1\sigma_0$.

First subcase: c_f is inserted. Assume that the procedure `refine-facet-or-edge` inserts c_f. Then the insertion radius of c_f is r_f and invariant (2) is preserved if

$$\beta_2 \geq 1. \tag{14}$$

It remains to guarantee the preservation of invariant (3). As in the case of rule R3, we bound the distance $\mathrm{d}(c_f, \mathrm{bd}\, S_k)$ and use lemma 5.1. Thus, invariant (3) is preserved if we can ensure that $\mathrm{d}(c_f, \mathrm{bd}\, S_k) \geq \frac{\alpha_1\sigma_0}{\eta_2}$. Let L_i be any curve segment in $\mathrm{bd}\, S_k$. As in the case of rule R3, we have:

$$\mathrm{d}(c_f, L_i) \geq r_f - 2\alpha_1\mu_0$$

Therefore, invariant (3) is satisfied if:

$$\alpha_2\sigma_0 - 2\alpha_1\mu_0 \geq \frac{\alpha_1\sigma_0}{\eta_2}$$

and

$$\beta_2\alpha_1\sigma_0 - 2\alpha_1\mu_0 \geq \frac{\alpha_1\sigma_0}{\eta_2}$$

which is

$$\frac{\alpha_2}{\alpha_1} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{15}$$

$$\beta_2 - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0}. \tag{16}$$

Second subcase: c_f is rejected. Assume that the procedure `refine-facet-or-edge` rejects the center c_f and inserts as new vertex a point c_e in a curve segment L_i. Point c_e is the center of the restricted Delaunay ball $B(c_e, r_e)$ of a segment $e \in \mathcal{D}_{|L_i}(\mathcal{P})$ and the ball $B(c_e, r_e)$ is encroached by c_f. The insertion radius of c_e is r_e. Let p be any vertex of e. Because p is not in $B(c_f, r_f)$ and both c_f and p belong to $B(c_e, r_e)$, we have

$r_f \leq \mathrm{d}(c_f, p) \leq 2r_e$.

Then,

$$r_e \geq \frac{r_f}{2} \geq \begin{cases} \frac{\alpha_2\sigma_0}{2} \\ \text{or} \\ \frac{\beta_2\alpha_1\sigma_0}{2}. \end{cases}$$

The invariant (2) is satisfied if:

$$\alpha_2 \geq 2\alpha_1 \tag{17}$$

and

$$\beta_2 \geq 2. \tag{18}$$

Because $B(c_e, r_e)$ include no vertex, if inequalities (17) and (18) hold, c_e is at least at distance $r_e \geq \alpha_1\sigma_0$ from $\mathrm{bd}\, L_i$. Then, lemma 5.1 implies then that invariant (3) is satisfied.

Rule R5 Assume that rule R5 is applied, and that c_t is the center of a tetrahedron t with a Delaunay ball $B(c_t, r_t)$ that either violates the size criteria (rule R5.1) or the shape criteria (rule R5.2).

The radius of insertion $r(c_t)$ of c_t is just r_t.

- If rule R5.1 is applied, we have $r_t \geq \sigma(c_t) \geq \sigma_0$.
- If rule R5.2 is applied, we have $r_t \geq \beta_3 l_{min}(t)$ where $l_{min}(t)$ is the length of the smallest edge of t. Let p be the last inserted vertex of the smallest edge of t. By induction, $l_{min}(t) \geq \alpha_1\sigma_0$. Thus $r_t \geq \beta_3\alpha_1\sigma_0$.

Rule R5. First subcase 5.1 Assume first that the procedure `refine-tet-or-facet-or-edge` inserts c_t as new vertex. The radius of insertion of c_t is r_t and invariant 2 is preserved if

$$\beta_3 \geq 1. \tag{19}$$

To ensure invariant (3), let y be the point on $\mathcal{S}$ closest to c_t. We have

$$\mathrm{d}(c_t, \mathcal{S}) = \mathrm{d}(c_t, y) \geq \mathrm{d}(c_t, \mathcal{P}') - \mathrm{d}(y, \mathcal{P}')$$

where $\mathcal{P}' = \mathcal{P} \cap \mathcal{F}_2$ is the current set of vertices on $\mathcal{F}_2$. Let S_k be the surface patch containing y and let q be the sample point in $\mathcal{P} \cap S_k$ closest to y. When rule R5 is applied, rule R4 no longer applies and therefore any restricted Delaunay ball $B(c,r)$ centered on S_k has a radius smaller than $\alpha_2\sigma(c)$. From the loose $\varepsilon-$sample lemma 4.10, we know then that any point p in S_k is at distance at most $\alpha_2(1+O(\alpha^2))\sigma(p)$ from the closest sample point. For small enough α_2, the constant $\alpha_2(1+O(\alpha^2))$ is less than $2\alpha_2$ and $\mathrm{d}(y,q) \leq 2\alpha_2\sigma(q)$. Thus,

$$\mathrm{d}(y,\mathcal{P}') \leq 2\alpha_2\sigma(q) \leq 2\alpha_2\mu_0.$$

and:

$$\begin{aligned}
\mathrm{d}(c_t,\mathcal{S}) = \mathrm{d}(c_t,y) &\geq \mathrm{d}(c_t,\mathcal{P}') - \mathrm{d}(y,\mathcal{P}') \\
&\geq r_t - 2\alpha_2\mu_0 \\
&\geq \begin{cases} \sigma_0 - 2\alpha_2\mu_0 \\ \text{or} \\ \beta_3\alpha_1\sigma_0 - 2\alpha_2\mu_0 \end{cases}
\end{aligned}$$

Hence, invariant (3) is satisfied if:

$$\begin{aligned}
\sigma_0 - 2\alpha_2\mu_0 &\geq \frac{\alpha_1}{\eta_3}\sigma_0 \\
\beta_3\alpha_1\sigma_0 - 2\alpha_2\mu_0 &\geq \frac{\alpha_1}{\eta_3}\sigma_0,
\end{aligned}$$

which is

$$1 - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_2\mu_0}{\sigma_0} \tag{20}$$

$$\beta_3\alpha_1 - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_2\mu_0}{\sigma_0}. \tag{21}$$

Rule R5. Second subcase 5.2. Assume next that the procedure `refine-tet-or-facet--or-edge` inserts the center c_f of a restricted Delaunay ball $B(c_f,r_f)$ circumscribed to a facet $f \in \mathcal{D}_{|S_k}$ encroached by c_t. The insertion radius of c_f is r_f. Let p be any vertex of the facet f. The Delaunay ball $B(c_t,r_t)$ circumscribing tetrahedron t does not include vertex p, thus $r_t \leq \mathrm{d}(c_t,p)$, and because c_t and p both belong to the ball $B(c_f,r_f)$, $\mathrm{d}(c_t,p) \leq 2r_f$. Therefore, $r_t \leq 2r_f$ and, according to which criteria R5.1 or **??** launched the rule R5, we get:

$$r_f \geq \frac{\sigma_0}{2}$$

or

$$r_f \geq \frac{\beta_3\alpha_1\sigma_0}{2}.$$

Hence, invariant (2) is satisfied if the following equalities holds

$$\frac{1}{2} \geq \alpha_1 \tag{22}$$

$$\beta_3 \geq 2. \tag{23}$$

To ensure that c_f satisfies invariant (3), we ensure that the distance $\mathrm{d}(c_f,\mathrm{bd}\,S_k)$ is at least $\frac{\alpha_1\sigma_0}{\eta_2}$ and apply lemma 5.1. Let $L_i \in \mathcal{L}$ be the edge closest to c_f on the boundary of $\S_k$. As in the case of rule R4, we have

$$\mathrm{d}(c_f,L_i) \geq r_f - 2\alpha_1\mu_0.$$

Then, invariant (3) is preserved if both following inequalities hold:

$$\frac{\sigma_0}{2} - 2\alpha_1\mu_0 \geq \frac{\alpha_1\sigma_0}{\eta_3}$$
$$\frac{\beta_3\alpha_1\sigma_0}{2} - 2\alpha_1\mu_0 \geq \frac{\alpha_1\sigma_0}{\eta_3}$$

which is

$$\frac{1}{2} - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_1\mu_0}{\sigma_0} \tag{24}$$
$$\frac{\beta_3}{2} - \frac{1}{\eta_3} \geq \frac{2\mu_0}{\sigma_0}. \tag{25}$$

Rule R5. Third subcase 5.3. We assume last that `refine-tet-or-facet-or-edge` inserts the center c_e of a restricted Delaunay ball $B(c_e, r_e)$ circumscribed to an edge $e \in \mathcal{D}_{|L_j}(\mathcal{P})$ for some $L_j \in \mathcal{L}$. The center c_e is inserted either because $B(c_e, r_e)$ is encroached by c_t (subcase 5.3.1) or because $B(c_e, r_e)$ is encroached by the center c_f of a surface Delaunay ball which is itself encroached by c_t (subcase 5.3.2). The insertion radius of c_e is r_e.

In subcase 5.3.1, we have:

$$r_e \geq \frac{\sigma_0}{2}$$

or

$$r_e \geq \frac{\beta_3\alpha_1\sigma_0}{2}.$$

The proof for Invariant (2) is exactly the same than in subcase 5.2, by just replacing $B(c_f, r_f)$ by $B(c_e, r_e)$ and considering a vertex p of the edge e circumscribed by $B(c_e, r_e)$. Thus Invariant (2) is satisfied if the inequalities (23) and t (22) hold.

In subcase 5.3.2, we have, as in subcase 5.2,

$$r_f \geq \frac{\sigma_0}{2}$$

or

$$r_f \geq \frac{\beta_3\alpha_1\sigma_0}{2}.$$

For any vertex p of e, $2r_e \geq \mathrm{d}(p, c_f) \geq r_f$, hence

$$r_e \geq \frac{\sigma_0}{4}$$

or

$$r_e \geq \frac{\beta_3\alpha_1\sigma_0}{4}.$$

Invariant (2) is preserved if both following inequalities hold :

$$\beta_3 \geq 4 \tag{26}$$
$$\frac{1}{4} \geq \alpha_1. \tag{27}$$

We still have to ensure invariant (3). Because $B(c_e, r_e)$ is a restricted Delaunay ball, hence empty of vertices, we know that $\mathrm{d}(c_e, \mathcal{Q}) \geq r_e$ which is greater than $\alpha_1\sigma_0$ if equation (26) and (27) holds. From lemma 5.1, this ensures that invariant invariant (3) is satisfied.

Summary of all conditions, and conclusion of the proof To sum up, the invariants (2) and (3) are maintained during the execution of the algorithm, if the following set of inequalities hold:

$$\alpha_1 \leq \alpha_2 \leq 1 \tag{28}$$
$$\alpha_1 \leq \eta_3 \leq \eta_2 \leq 1 \tag{29}$$

(rule R3)

(case $p \in \mathcal{P} \setminus \bigcup \mathcal{F}_2$)

$$\frac{1}{\eta_3} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{8}$$

(case $p \in \bigcup \mathcal{F}_2$, not interrelated to c_f)

$$\frac{1}{\alpha_1} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{9}$$

(case $p \in \bigcup \mathcal{F}_2$, interrelated to c_f)

$$\frac{1}{\eta_2} \geq 2(\sqrt{2}+1)\frac{\mu_0}{\sigma_0} \tag{13}$$

(rule R4)

(case c_f inserted)

$$\beta_2 \geq 1 \tag{14}$$
$$\frac{\alpha_2}{\alpha_1} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{15}$$
$$\beta_2 - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{16}$$

(case c_f rejected, c_e inserted)

$$\alpha_2 \geq 2\alpha_1 \tag{17}$$
$$\beta_2 \geq 2 \tag{18}$$

(rule R5)

(first subcase, the center of a tetrahedron c_t is inserted)

$$\beta_3 \geq 1 \tag{19}$$
$$1 - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_2\mu_0}{\sigma_0} \tag{20}$$
$$\beta_3\alpha_1 - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_2\mu_0}{\sigma_0} \tag{21}$$

(second subcase, the center c_f of a restricted Delaunay facet is inserted)

$$\beta_3 \geq 2 \tag{23}$$
$$\frac{1}{2} \geq \alpha_1 \tag{22}$$
$$\frac{1}{2} - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_1\mu_0}{\sigma_0} \tag{24}$$
$$\frac{\beta_3}{2} - \frac{1}{\eta_3} \geq \frac{2\mu_0}{\sigma_0} \tag{25}$$

(third subcase, the center c_e of a restricted Delaunay edge is inserted.)

$$\beta_3 \geq 4 \tag{26}$$

$$\frac{1}{4} \geq \alpha_1. \tag{27}$$

This set of conditions can be reduced. First, condition (22) is weaker than (27), (14) is weaker than (18) and (19) and (23) are weaker than (26). Because $\alpha_2 \leq 1$, equation (9) is weaker than (15). Because $\mu_0 \geq \sigma_0$ equation (17) is weaker than (15), condition (18) is weaker than (16), condition (26) is weaker than (25) and condition (27) is weaker than (24).

Thus, the set of conditions reduces to:

$$\alpha_1 \leq \alpha_2 \leq 1 \tag{??}$$

$$\alpha_1 \leq \eta_3 \leq \eta_2 \leq 1 \tag{??}$$

$$\frac{1}{\eta_3} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{8}$$

$$\frac{1}{\eta_2} \geq 2(\sqrt{2}+1)\frac{\mu_0}{\sigma_0} \tag{13}$$

$$\frac{\alpha_2}{\alpha_1} - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{15}$$

$$\beta_2 - \frac{1}{\eta_2} \geq \frac{2\mu_0}{\sigma_0} \tag{16}$$

$$1 - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_2\mu_0}{\sigma_0} \tag{20}$$

$$\beta_3\alpha_1 - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_2\mu_0}{\sigma_0} \tag{21}$$

$$\frac{1}{2} - \frac{\alpha_1}{\eta_3} \geq \frac{2\alpha_1\mu_0}{\sigma_0} \tag{24}$$

$$\frac{\beta_3}{2} - \frac{1}{\eta_3} \geq \frac{2\mu_0}{\sigma_0}. \tag{25}$$

This set of condition is satisfied by the following choice which maximizes the values of η_2, η_3, α_1 and α_2 and minimizes the choice of β_2 and β_3.

$$\frac{1}{\eta_2} = (\sqrt{2}+1)\frac{2\mu_0}{\sigma_0} \tag{30}$$

$$\frac{1}{\eta_3} = (\sqrt{2}+2)\frac{2\mu_0}{\sigma_0} \tag{31}$$

$$\beta_2 = (\sqrt{2}+2)\frac{2\mu_0}{\sigma_0} \tag{32}$$

$$\alpha_2 = \frac{1}{\frac{2\mu_0}{\sigma_0}+1} \tag{33}$$

$$\alpha_1 = \frac{\alpha_2}{(\sqrt{2}+2)\frac{2\mu_0}{\sigma_0}} = \frac{1}{(\sqrt{2}+2)\frac{2\mu_0}{\sigma_0}\left(\frac{2\mu_0}{\sigma_0}+1\right)} \tag{34}$$

$$\beta_3 = \frac{1}{\alpha_1} = (\sqrt{2}+2)\frac{2\mu_0}{\sigma_0}\left(\frac{2\mu_0}{\sigma_0}+1\right). \tag{35}$$

□

These values ensure that invariants (2) and (3) are maintained. Therefore each new vertex is inserted with an insertion radius lower bounded by $\alpha_1\sigma_0$. The standard volume argument shows that the algorithm terminates.

6 Implementation and results

The algorithm has been implemented in C++, using the library CGAL [CGAL], which provided us with an efficient and flexible implementation of the three-dimensional Delaunay triangulation. Once the Delaunay refinement process described above is over, a sliver exudation [CDE+00] step is performed. This step does not move or add any vertex but it modifies the mesh by switching the triangulation into a weighted Delaunay triangulation with carefully chosen weights. As in [ORY05] the weight of each vertex in the mesh is chosen in turn so as to maximize the smallest dihedral angle of any tetrahedron incident to that vertex while preserving in the mesh any facet that belongs to the restricted triangulation of an input surface patch.

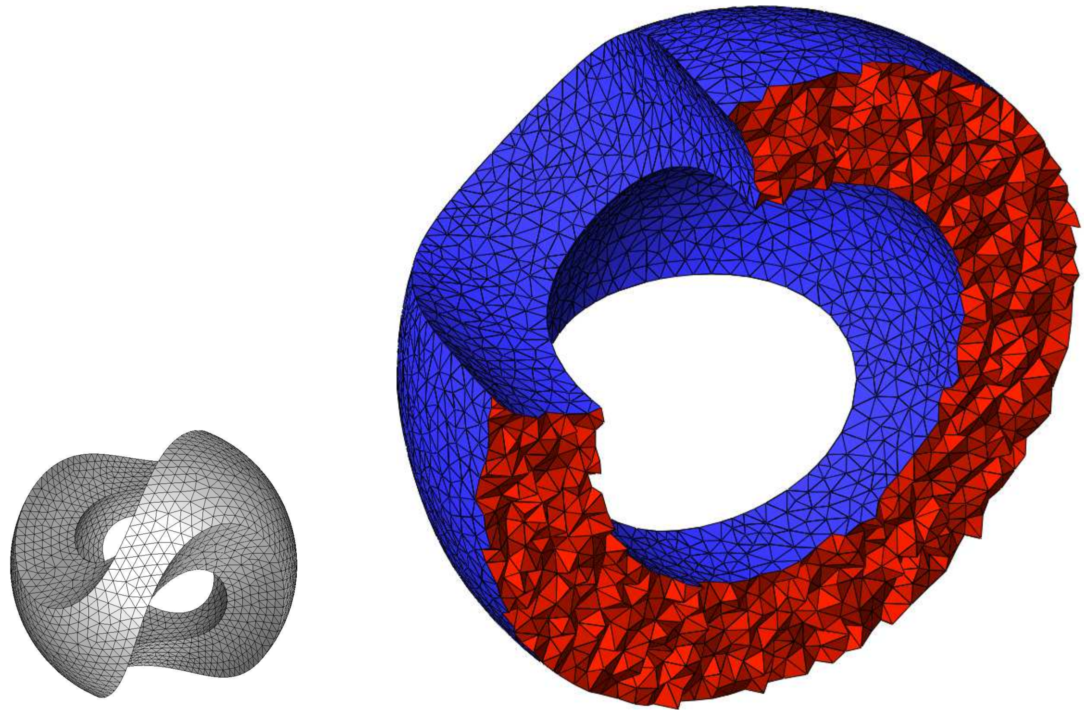

Figure 3: **Sculpt model.** On the left: the input surface mesh. On the right: the output mesh (blue: facets of the surface mesh, red: tetrahedra of the volume mesh that intersect a given plan). The mesh counts 22923 tetrahedra.

Our mesh generation algorithm interacts with the input curve segments and surface patches through an oracle that is able to detect and compute intersections between planar triangles and curve segments and between straight segments and surface patches. Currently, we have only one implementation of such an oracle which handles input curve segments and surface patches described as respectively as polylines and triangular meshes. Thus, with respect to input features our algorithm currently act as a remesher, but this is not a limitation of the method.

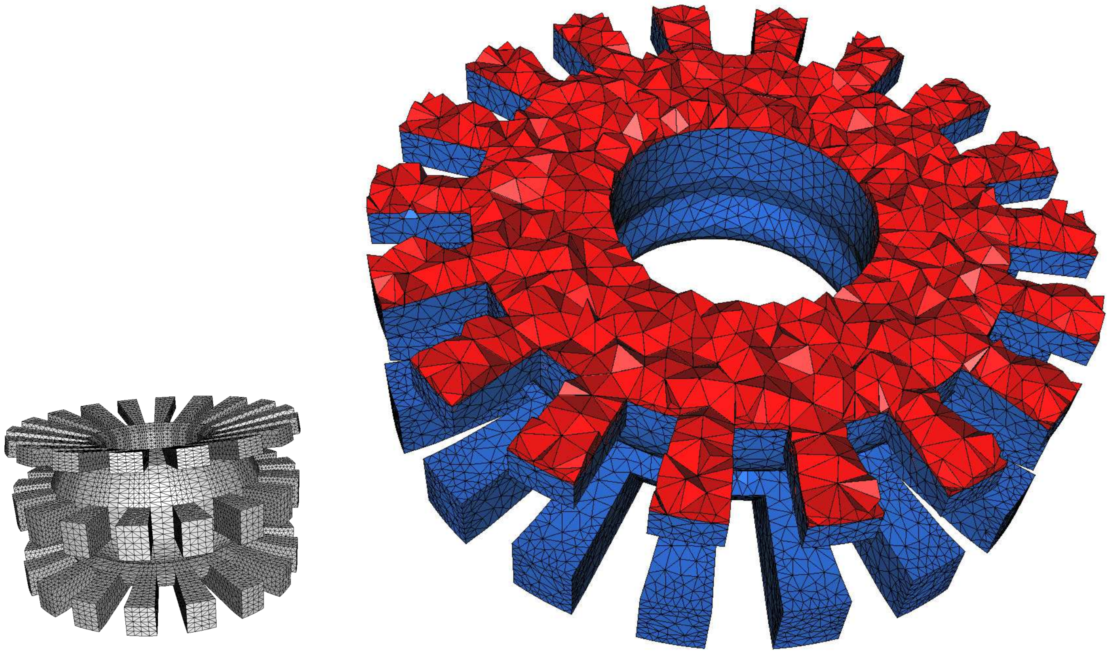

Figure 4: **ITER model.** A mesh with 72578 tetrahedra.

The figure 3 shows a mesh generated by our algorithm. The input surface of the figure 3 is made of six curved edges, and four surface patches. The input surface was given by a surface mesh. We have considered that an edge of the mesh is a sub-segment of a curved segment of the surface when the normal deviation at the edge is greater than 60 degree. The curved segments and surfaces patches of the input surface are nicely approximated, as well as the normals (the result surface of figure 3 has been drawn without any OpenGL smoothing). Each element of the result mesh has a Delaunay ball smaller than a given sizing field. In the figure 3, the sizing field has been chosen uniform. After sliver exudation, the worst tetrahedra in the mesh has a dihedral angle of 1.6 degree.

Another example is shown on Figure 4 where a mechanical piece which is part of the International Thermonuclear Experimental Reactor (ITER) has been meshed.

7 Conclusion and future work

The algorithm provided in this paper is able to mesh volumes bounded by piecewise smooth surfaces. The output mesh has guaranteed quality and its granularity adapts to a user defined sizing field. The boundary surfaces and their sharp 1-dimensional features are accurately and homeomorphically represented in the mesh. The main drawback of the algorithm is the restriction imposed on dihedral angles made by tangent planes on singular points. Small angles are known to trigger an ever looping of Delaunay refinement algorithm. The main idea to handle this problem is to define a protected zone around sharp features where the Delaunay refinement is restricted to prevent looping. This strategy assumes that the mesh already includes restricted Delaunay submeshes homeomorph to the input surface patches and curved segments. This could be achieved using a strategy analog to the strategy proposed to conform Delaunay triangulation [MMG00, CCY04]. Another promising way to protect sharp feature which is proposed by [CDR07] is to use weighted Delaunay triangulation with weighted points on sharp features.

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06902 Sophia Antipolis Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399